Lecture note for
ASLS
accelerator School

SR monitor lecture 4, 5 and 6

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Lecture 4

Simple introduction to quantum optics and coherence of light

Flavor of Quantum Optics

Recently, many peoples use the word "photon"

"coherence"

as photon beam, photon physics….

coherence, coherent…..etc.

But almost nobody knows what is

physical of "photon" and coherence.

Following recent trends in Quantum optics,

<u>we like to not use the word “photon”.</u>

I try to introduce introductive idea of Quantum optics and the word “photon” and “coherence” in this chapter.

Quantization of the electro-magnetic field and the photon number state

The physics on wave and particle aspects of light is one of big issue in the optical physics from the time of Newton.

quantization of the electro-magnetic field by using well-known procedure for quantize the harmonic oscillator.

Using this procedure, the electro-magnetic field is represented by creation, annihilation operators and photon number state.

Let consider the Maxwell equations written with current $\boldsymbol{J}$ and charge ρ in the vacuum.

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

Using Coulomb gage,

$$\nabla \cdot \boldsymbol{A}' = 0$$

In the free field which has no charge r and no current $\boldsymbol{J}$, the Maxwell equation will given by,

$$\nabla^2 \boldsymbol{A} - \frac{1}{c} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

$$\nabla \cdot \boldsymbol{A} = 0$$

$$\phi = 0$$

$\boldsymbol{E}$ and $\boldsymbol{B}$ are given by

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\varepsilon_0} = -\frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} = \nabla \boldsymbol{A}$$

Integrated energy in the optical resonator is given by

$$\mathcal{H} = \frac{1}{2}\int (\varepsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2)\, d^3 r$$

$$= \frac{1}{2}\int \left\{ \varepsilon_0 \left(\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{t}} \right)^2 + \frac{1}{\mu_0} \left(\nabla \times \boldsymbol{A} \right)^2 \right\} d^3 r$$

Representing the solution of the Maxwell equation in the Optical resonator by the product of temporal and spatial parts

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{1}{\sqrt{\varepsilon_0}}\sum_{l} q_l(t)\boldsymbol{u}_{\boldsymbol{l}}(\boldsymbol{r})$$

$$\mathrm{E}(\mathrm{r},\mathrm{t})=-\frac{1}{\sqrt{\boldsymbol{\varepsilon}_0}}\sum_{\mathrm{l}}\dot{\mathrm{q}}_{\mathrm{l}}(\mathrm{t})\mathrm{u}_{\mathrm{l}}(\mathrm{r})$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\frac{1}{\mu_0\sqrt{\varepsilon_0}}\sum_{l} q_l(t)\nabla\times\boldsymbol{u}_{\boldsymbol{l}}(\boldsymbol{r})$$

Substituting $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{H}$ in $\mathscr{H}$ by

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{1}{\sqrt{\varepsilon_0}}\sum_{l} q_l(t)\boldsymbol{u}_l(\boldsymbol{r})$$

$$\mathrm{E}(\mathrm{r},\mathrm{t})=-\frac{1}{\sqrt{\varepsilon_0}}\sum_{\mathrm{l}} \dot{\mathrm{q}}_{\mathrm{l}}(\mathrm{t})\mathrm{u}_{\mathrm{l}}(\mathrm{r})$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\frac{1}{\mu_0\sqrt{\varepsilon_0}}\sum_{l} q_l(t)\nabla\times\boldsymbol{u}_l(\boldsymbol{r})$$

We obtain

$$\mathcal{H} = \frac{1}{2}\sum_{l}\left(\dot{q}_l^{\,2} + \boldsymbol{\omega}_l^2\, q_l^2 \right)$$

Remember Hamiltonian for harmonic oscillator

$$\mathcal{H} = \frac{m}{2}\left(\dot{x}^{2} + \boldsymbol{\omega}\, x^{2} \right)$$

This representation is same form as in the simple harmonic oscillator which has the mass m=1

We can conclude the electro-magnetic field is equal to the infinite assembly of independent oscillators.

The Hamilton equation is generally written by

$$\frac{\partial \mathcal{H}_l}{\partial q_l} = -\dot{p}_1 \qquad \frac{\partial \mathcal{H}_l}{\partial p_l} = \dot{q}_1$$

$$\begin{cases} q_l = \sqrt{\dfrac{\hbar}{2\omega_l}}\left(a^*_l + a_l\right) \\ p_l = i\sqrt{\dfrac{\hbar\omega_l}{2}}\left(a^*_l - a_l\right) \end{cases}$$

the coefficients are decided the $\mathcal{H}$ can measure with the unit for energy of photon

Using a_l and a^*_l, the $\mathcal{H}$ is then given by

$$\mathcal{H} = \frac{1}{2}\sum_l \hbar\,\omega_l\left(a^*_l\, a_l + a_l\ \ a^*_l\ \right)$$

a_l and a^*_l are the classical number (c-number), and a_l and a^*_l are commutable. Substituting q_l and p_l and solving a_l and a^*_l

$$\begin{cases} a_l\left(t\right) = a_l\left(0\right)e^{-i\omega_l \boldsymbol{t}} \\ a^*_l\left(t\right) = a^*_l\left(0\right)e^{i\omega_l \boldsymbol{t}} \end{cases}$$

Then, p_l and q_l becomes

$$\begin{cases} q_l(t) = \sqrt{\dfrac{\hbar}{2\omega_l}}\left\{a^*_l(0)e^{i\omega_l t} + a_l(0)e^{-i\omega_l t}\right\} \\ p_l(t) = \sqrt{\dfrac{\hbar\omega_l}{2}}\left\{a^*_l(0)e^{i\omega_l t} - a_l(0)e^{-i\omega_l t}\right\} \end{cases}$$

we obtain representation for the electro-magnetic field in the optical resonator as $a_l(0)$ for plus frequency component and $a^*_l(0)$ for minas frequency components

$$\boldsymbol{A}(\boldsymbol{r},t)=\sum_{\boldsymbol{l}}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_{\boldsymbol{l}}}}\left\{a^*_l(0)e^{i\omega_l \boldsymbol{t}}+a_l(0)e^{-i\omega_l \boldsymbol{t}}\right\}\boldsymbol{u_l}(\boldsymbol{r})$$

$$\boldsymbol{E}(\boldsymbol{r},t)=\sum_{\boldsymbol{l}} i\sqrt{\frac{\hbar\omega_{\boldsymbol{l}}}{2\varepsilon_0}}\left\{a^*_l(0)e^{i\omega_l \boldsymbol{t}}-a_l(0)e^{-i\omega_l \boldsymbol{t}}\right\}\boldsymbol{u_l}(\boldsymbol{r})$$

In the same manner, we can describe traveling wave under periodical boundary condition

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t)=\sum_{\boldsymbol{l}}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_{\boldsymbol{l}}}}\left\{\hat{a}_l(0)e^{-i\omega_l \boldsymbol{t}}+\hat{a}^{\dagger}(0)e^{i\omega_l \boldsymbol{t}}\right\}\boldsymbol{u}_{\boldsymbol{l}}(\boldsymbol{r})$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t)=\sum_{\boldsymbol{l}} i\sqrt{\frac{\hbar\omega_{\boldsymbol{l}}}{2\varepsilon_0}}\left\{\hat{a}_l(0)e^{-i\omega_l \boldsymbol{t}}+\hat{a}^{\dagger}(0)e^{i\omega_l \boldsymbol{t}}\right\}\boldsymbol{u}_{\boldsymbol{l}}(\boldsymbol{r})$$

Quantize of the electro-magnetic field

Explanation for the particle aspect of light, we need to quantize the electro-magnetic field.

we can apply the method for quantizing the harmonic oscillator to quantize the electro-magnetic field. Let introduce the Operators $\hat{q}_l$ and $\hat{p}_l$ by quantizing q_l and p_l .

$$\left[\hat{q}_l , \hat{p}_l\right] = i\hbar\delta_{ll'}$$

The Hamiltonian of electro-magnetic field becomes

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{l}\left(\dot{\hat{q}}_l^{\,2} + \omega_l^2\, \hat{q}_l^2\right) = \frac{1}{2}\sum_{l}\left(\hat{p}_l^2 + \omega_l^2\, \hat{q}_l^2\right)$$

The operators are introduced instead of a_l and a^*_l, then the Operators $\hat{q}_l$ and $\hat{p}_l$ are given by

$$\begin{cases} \hat{q}_l = \sqrt{\dfrac{\hbar}{2\omega_l}}\left(\hat{a}^\dagger{}_l + \hat{a}_l\right) \\[2em] \hat{p}_l = \sqrt{\dfrac{\hbar}{2\omega_l}}\left(\hat{a}^\dagger{}_l - \hat{a}_l\right) \end{cases}$$

Using , $\hat{a}_l$ *and* $\hat{a}^{\dagger}{}_l$ the Hamiltonian becomes

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{\boldsymbol{l}} \hbar\omega_{\boldsymbol{l}}\left(\hat{a}^{\dagger}{}_l \; \hat{a}_l + \frac{1}{2}\right)$$

Let E_n and $\left|n\right\rangle$ (n=0,1,2,.....) be the eigenvalue and the eigenfunction of the Hamiltonian, respectively. The Hamiltonian equation is given by,

$$\hat{\mathcal{H}}\left|n\right\rangle = \frac{1}{2}\sum_{\boldsymbol{l}} \hbar\omega_{\boldsymbol{l}}\left(\hat{a}^{\dagger}{}_l \; \hat{a}_l + \frac{1}{2}\right)\left|n\right\rangle = E_n\left|n\right\rangle$$

The energy corresponding to *n* is,

$$E_n = \left(n + \frac{1}{2} \right) \hbar \omega$$

This equation gives interesting aspect for number *n*=0.   It is no photon in the system,

But still system has the energy of

$$E_0 = \frac{1}{2} \hbar \omega$$

This called vacuum energy

Let us consider the expectation of $\hat{a}^{\dagger}\ \hat{a}$

$$\langle n|\hat{a}^{\dagger}\ \hat{a}|n\rangle = n$$

$|n\rangle$ represents the state which is *n* photons are exist as oscillators in the state.

Let us introduce the Number operator as,

$$\hat{n} \equiv \hat{a}^{\dagger}\ \hat{a}$$

The eigenvalue of this operator is given by

$$\langle n|\hat{n}|n\rangle = n$$

Which indicates number of photon.

Fluctuation of photon number

$$\langle \hat{n} \rangle = \langle n | \hat{n} | n \rangle = n$$

$$\langle \hat{n}^2 \rangle = \langle n | \hat{n}^2 | n \rangle = n^2$$

$$\left(\Delta n^2 \right) \equiv \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = 0$$

This result means photon number state is the state having obvious number of photon to *n* and no fluctuation in photon number!

Fluctuation of the electric field

$$\left\langle \hat{E} \right\rangle = \langle n | \hat{E} | n \rangle = 0$$

$$\left\langle \hat{E}^2 \right\rangle = \frac{\hbar\omega}{\varepsilon_0 V} \langle n | \ \hat{a}^\dagger \hat{a} + \frac{1}{2} | n \rangle = \frac{\hbar\omega}{\varepsilon_0 V} \left(n + \frac{1}{2} \right)$$

then,

$$\Delta E = \left(\left\langle \hat{E}^2 \right\rangle - \left\langle \hat{E} \right\rangle^2 \right)^{\frac{1}{2}} = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(n + \frac{1}{2} \right)^{\frac{1}{2}}$$

Important! In Photon number state, expectation of the electric field is 0!

<u>It is the particle (Photon) aspect of the light.</u>

Phase operator and phase state

In Photon number state, expectation of the electric field is 0, and it is representation of the particles aspect of the light.

The oscillating electro-magnetic field is represented by the amplitude and the phase. The information of amplitude is given by photon number.

In other hand, How we can represent phase with the quantum physics?

Dirac applied Poisson bracket equation of motion $\{\phi, \mathcal{H}\} = d\phi / dt = -\omega$ in classical theory to this system, i.e.

$$i\hbar\left\{ \phi , \mathcal{H} \right\} = -i$$

He also assumes following commutator in the quantum theory by corresponding relation

$$\left\{ \hat{\phi}_{\boldsymbol{D}} , \hat{n} \right\} = -i$$

He assumed $\hat{\phi}_{\boldsymbol{D}}$ is Hermitian operator.

The range measurement using the interference of the light is one of most precise measurement. Which limit exists in phase measurement in there from the view point of quantum physics?

To answer for these questions, we must consider following fundamental problems.

Can we find any Hermitian phase operator corresponding to measurement for the phase? Can we identify the phase which is defined by such phase operator, as the same physical meaning of phase as in the classical theory?

When $\hat{\phi}_D$ is Hermitian operator, following to the definition of the exponent function represented by power series expansion, there exist an unitary exponent operator as follows.

$$\exp\left(\pm i\hat{\phi}_D\right)$$

Using this Dirac's Idea,

we can represent annihilation operator $\hat{a}$ by the product of an amplitude $\hat{n}^{1/2}$ and a phaser $\exp\left(i\hat{\phi}_D\right)$ as follows

$$\hat{a} = \mathbf{exp}\left(i\hat{\phi}_D\right)\hat{n}^{1/2}$$

Perhaps we can write following phase state and the eigenvalue of phase

$$\mathbf{exp}\left(\pm i\hat{\phi}_{\boldsymbol{D}}\right)\left|\phi_m\right\rangle = \mathbf{exp}\left(\pm i\phi_m\right)\left|\phi_m\right\rangle$$

But big problem is $\hat{\phi}_{\boldsymbol{D}}$ is not Hermitian operator.

We also must to chek Dirc's assumes following commutator in the quantum theory by corresponding relation

$$\left\{ \hat{\phi}_{\boldsymbol{D}} \, , \hat{n} \right\} = \hat{\phi}_{\boldsymbol{D}} \hat{n} - \hat{n} \hat{\phi}_{\boldsymbol{D}} = -\boldsymbol{i}$$

To check the Dirac's phase operator with matrix form checking n , n' element of matrix,

$$(n' - n)\langle n|\,\hat{\phi}_D\,|n'\rangle = -i\delta_{n\,,n'} \neq i$$

It is a mistake to put right hand term to scalar quantity .

We recognized that considering the phase with quantum physics is not easy.

Susskind and Glogower introduced following Hermitian sine and cosine operators by a summation and subtraction of exponent functions,

$$\widehat{\mathbf{cos}}_{SG}(\phi) = \left\{\widehat{\mathbf{exp}}_{SG}(i\phi) + \widehat{\mathbf{exp}}_{SG}(-i\phi)\right\}/2$$

$$\widehat{\mathbf{sin}}_{SG}(\phi) = \left\{\widehat{\mathbf{exp}}_{SG}(i\phi) - \widehat{\mathbf{exp}}_{SG}(-i\phi)\right\}/2i$$

Otherwise Pegg anf Bernett introduce another type of Phase operator, but…………

Coherent state

Let $|\alpha\rangle$ and α are eigenstate and eigenvalue of annihilation operator

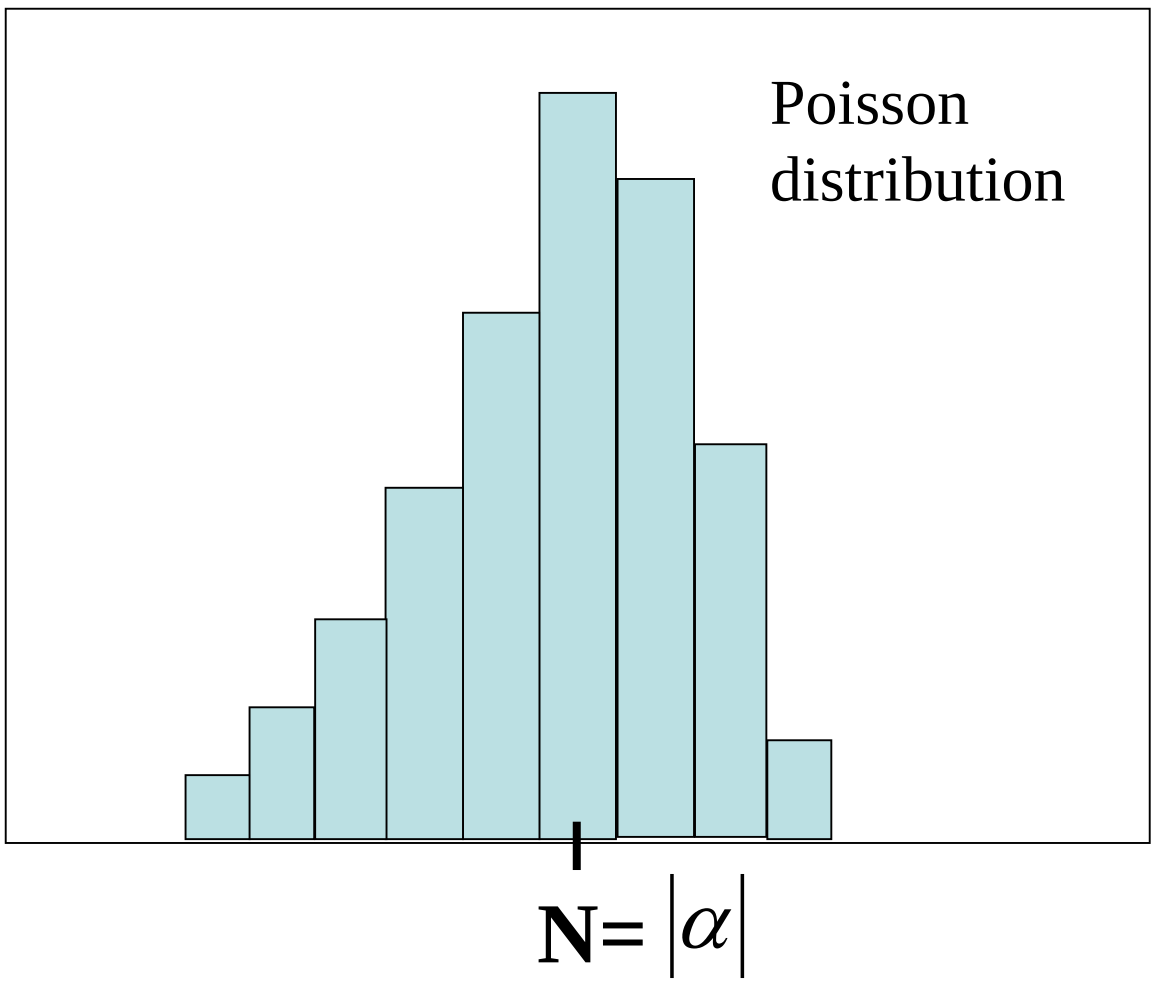
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

The probability distribution of state $|n\rangle$ at $|\alpha\rangle$ is given by

$$\left|\langle n|\alpha\rangle\right|^2 = \mathbf{exp}\left(-|\alpha|^2\right)\frac{|\alpha|^{2n}}{n!}$$

This probability distribution is called by the Poisson distribution with a average value of $|\alpha|^2$.


Poisson
distribution
$\mathbf{N} = |\alpha|$

Return to the electric field,

$$\hat{E}(t,z) = i\varepsilon \left\{ \hat{a}\, e^{-i(\omega t - kz)} - \hat{a}^{\dagger} e^{i(\omega t - kz)} \right\}$$

Replacing $\hat{a}$ and $\hat{a}^{\dagger}$ those are represented by using the real numbers $\hat{x}_1$ and $\hat{x}_2$ as follows ,

$$\hat{a} \equiv \hat{x}_1 + i\,\hat{x}_2 \qquad \hat{a}^{\dagger} \equiv \hat{x}_1 - i\,\hat{x}_2$$

Then the electric field becomes,

$$\hat{E}(t,z) = 2\varepsilon \left\{ \hat{x}_1 \,\mathbf{sin}(\omega t - kz) - \hat{x}_2 \,\mathbf{cos}(\omega t - kz) \right\}$$

The expectation for the electric field at coherent state $|\alpha\rangle$ is ,

$$\langle\alpha|\ \hat{E}(t,z)|\alpha\rangle = i\varepsilon\left\{\alpha\,\mathbf{exp}(-i\omega t + i\,k\,z) - \alpha^{*}\,\mathbf{exp}(i\,\omega t + i\,k\,z)\right\}$$

$$= 2\varepsilon|\alpha|\,\mathbf{sin}(\omega t - k\,z - \theta)$$

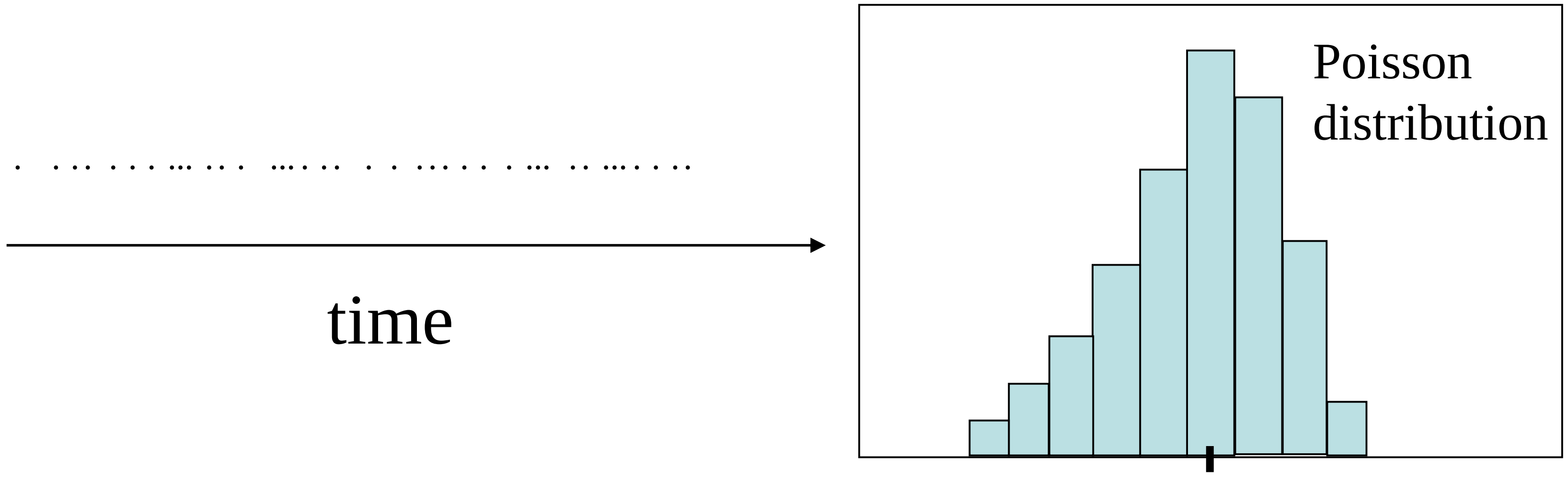
Therefore, amplitude of the electric field is $2\varepsilon|\alpha|$, and it's phase is θ.

And the expectation of square of the electric field at coherent state $|\alpha\rangle$ is ,

$$\langle\alpha|\ \hat{E}^{\,2}(t,z)|\alpha\rangle$$
$$= -\varepsilon^2 \left\{\alpha^2 \,\mathbf{exp}(-2\,i\omega\,t + 2\,i\,k\,z) - 2\alpha^{*}\alpha - 1 + \alpha^{*2}\,\mathbf{exp}(2\,i\omega\,t - 2\,i\,k\,z)\right\}$$
$$= \varepsilon^2 \left\{4|\alpha|^2\ \mathbf{sin}^2(\omega\,t - k\,z - \theta) + 1\right\}$$

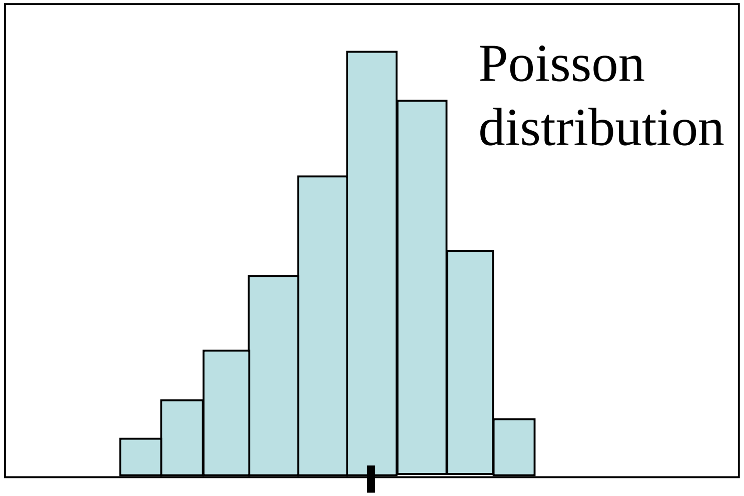
The last 1 in this equation is come from the commutation relation, and it is indicated that the expectation of E^2 is not simply square of the E.

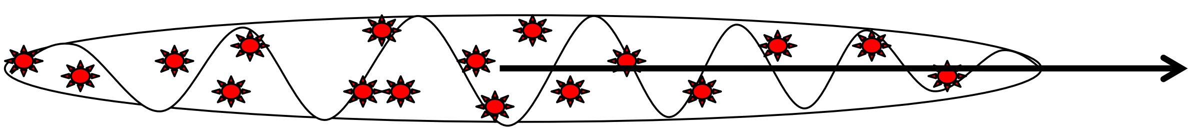
The coherent state : the provability to find the photon at any time t is random



$\mathbf{N}=|\alpha|$

The coherent state : the provability to find the photon at any time t is random





$$\mathbf{N}=|\alpha|$$

1-1. Introduction to coherence of light

Coherence : nth order correlation function of (normalized) field E_n of light mode ψ_n at position ant time r_{2n},t_{2n}.

$$g^n(\boldsymbol{r}_1,\boldsymbol{t}_1......\boldsymbol{r}_{2n},\boldsymbol{t}_{2n})=\left\langle \boldsymbol{E}_1(\boldsymbol{r}_1,\boldsymbol{t}_1).....\boldsymbol{E}_{2n}(\boldsymbol{r}_{2n},\boldsymbol{t}_{2n})\boldsymbol{E}_{2n}{}^*(\boldsymbol{r}_{2n},\boldsymbol{t}_{2n}).....\boldsymbol{E}_1{}^*(\boldsymbol{r}_1,\boldsymbol{t}_1)\right\rangle$$

Coherence is how cohere

1st order : field 1 photon in system.

2ed order : intensity 2 photons in system.

3ed order : 3 photons in system

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.
.
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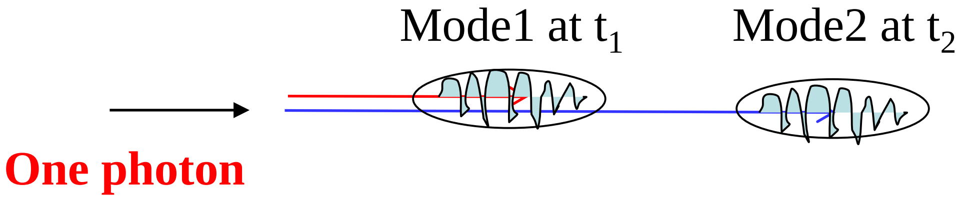
n-th order: n photons in system

Simple understanding of 1st and 2ed order coherence of the light

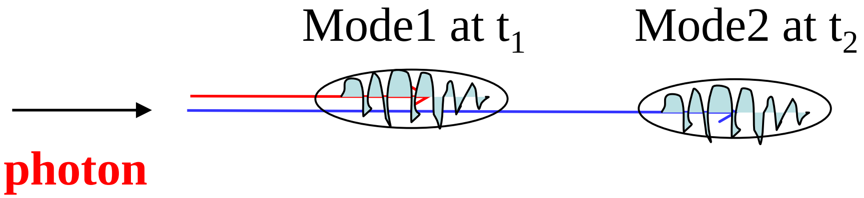
1-1-a. 1st order temporal coherence of light

Consider tow modes of one photon come in the different time



How cohere the two modes 1 $\psi_1(x,t_1)$ and mode2 $\psi_2(x,t_2)$ of one photon (N photons of single mode) as the time ?

Consider tow modes of one photon come in the different time



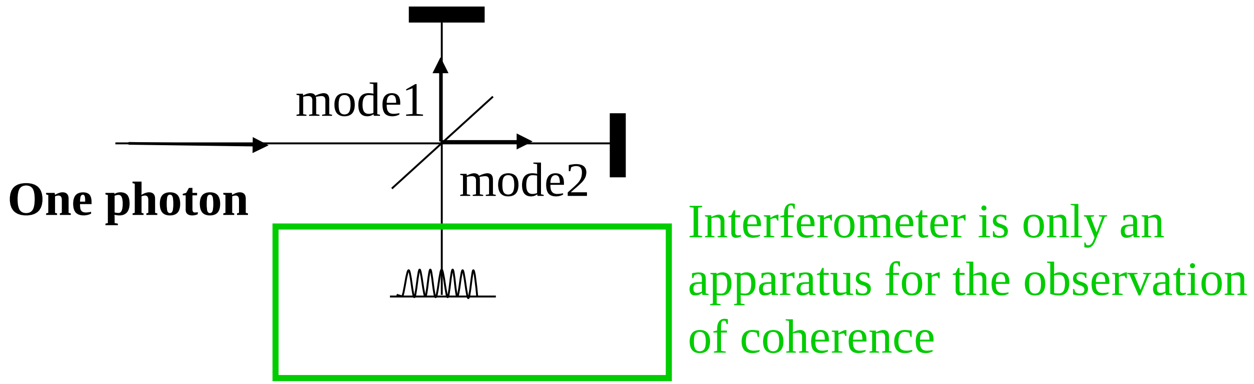
How cohere the two modes 1 $\psi_1(x,t_1)$ and mode2 $\psi_2(x,t_2)$ of

one photon (N photons of single mode) as the time ?

$$\left\langle E_1(t_1)E^*{}_2(t_2)\right\rangle = \int E_1(x,t)E^*{}_2(x,t+t')dt = 1$$ **1st order temporal coherent**

$$=0$$ **1st order temporal incoherent**

Observation of 1st order temporal coherence with intensity division type Interferometer.



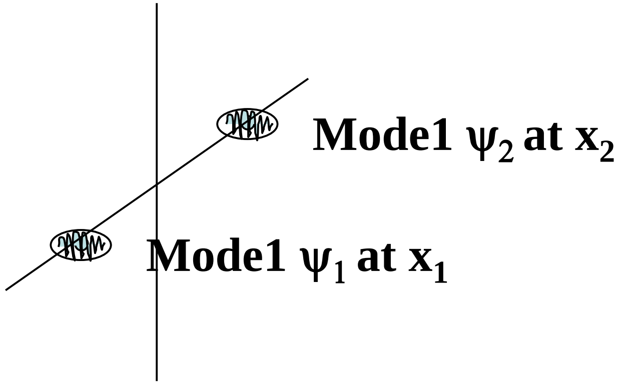
To observe contrast of interference fringe,

Contrast=1 : 1st order temporal coherent

Contrast=0 : 1st order temporal incoherent

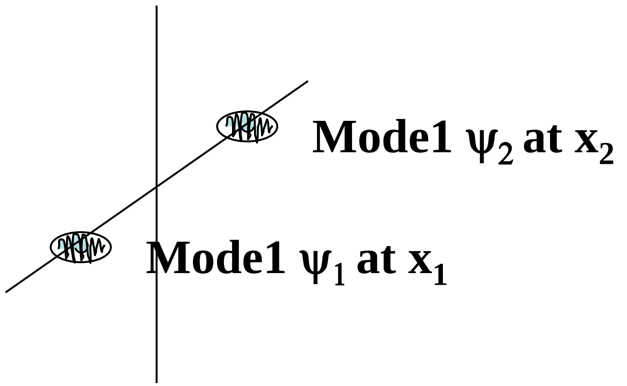
1-1-b. 1st order spatial coherence of the light

Consider two modes of one photon came at different spatial positions.



How cohere the two modes 1 $\psi_1(x_1,t)$ and mode2 $\psi_2(x_2,t)$ of <u>one photon</u> (N photons in single mode) as the position ?

Consider two modes of one photon came at different spatial positions.

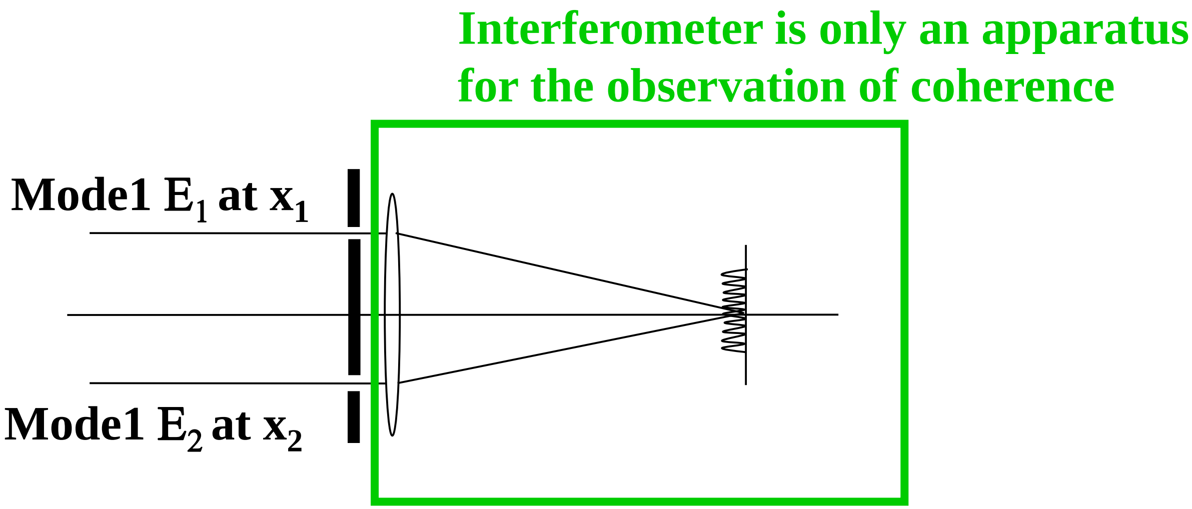


How cohere the two modes 1 $\psi_1(x_1,t)$ and mode2 $\psi_2(x_2,t)$ of one photon ψ (N photons in single mode ψ_n) as the position ?

$$\langle E_1(x_1)E^*{}_2(x_2)\rangle = \int E_1(x,t)E^*{}_2(x+x',t)dx = 1$$ **1st order spatial coherent**

$$=0$$ **1st order spatial incoherent**

Observation of 1^{st} order spatial coherence with wavefront division type interferometer



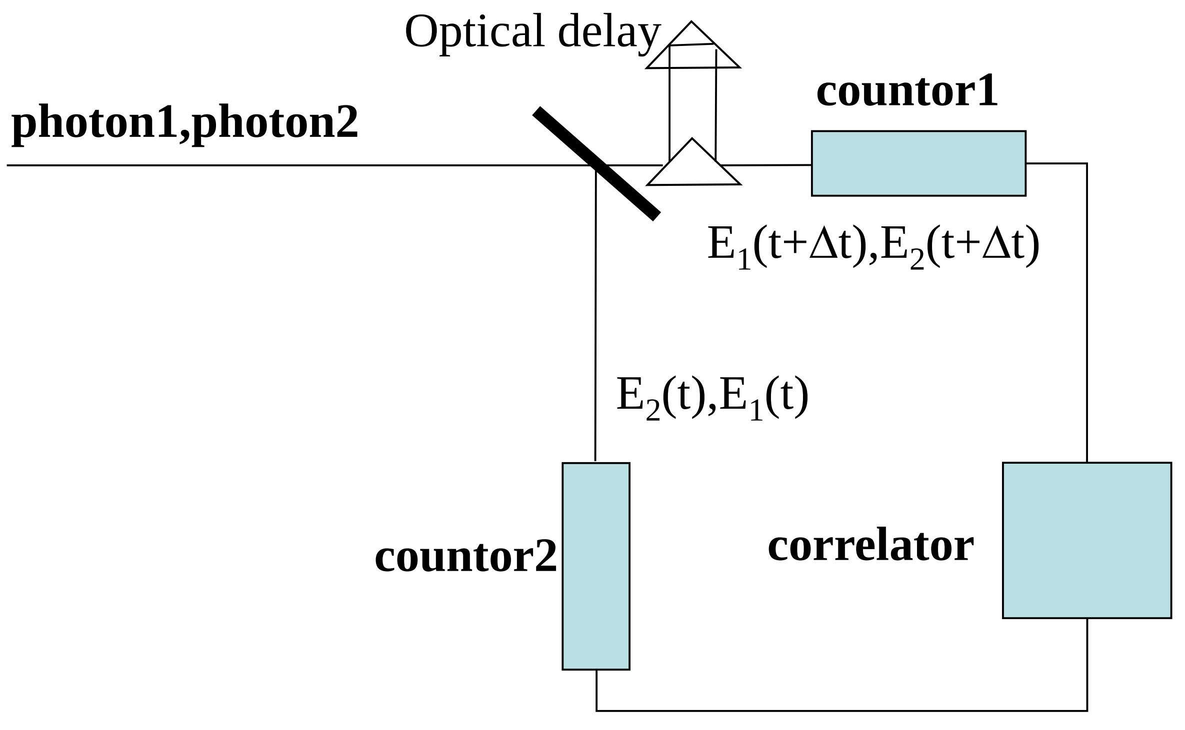
To observe contrast of interference fringe,

Contrast=1 : 1^{st} order spatial coherent

Contrast=0 : 1^{st} order spatial incoherent

1-1-c. 2^{ed} order temporal coherence of the light

Consider two photons come in the different time.

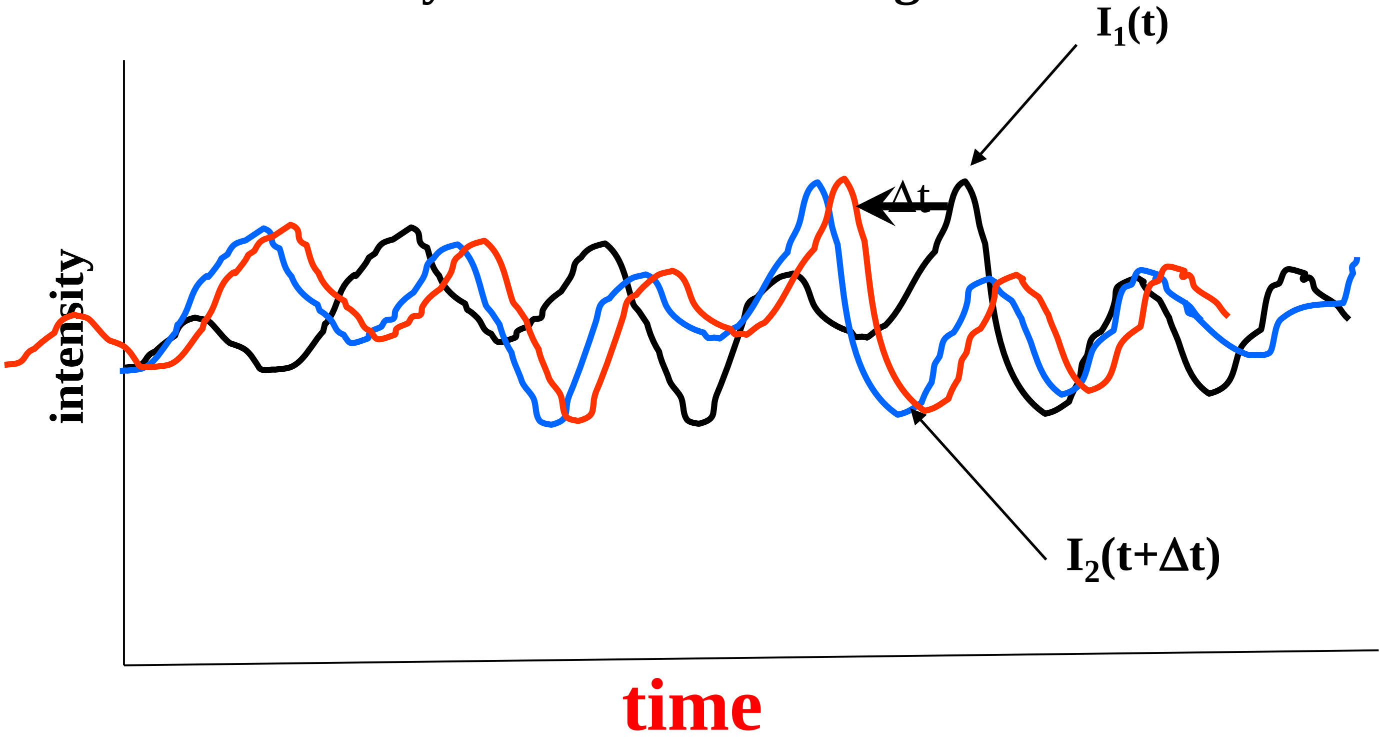


Coincidence count in time t and t+Δt is given by;

$$P_{12}(t, t+\Delta t) = \langle E_1(t) E_2(t+\Delta t) E^*{}_2(t) E^*{}_1(t+\Delta t) \rangle$$

What is the physical meaning of P_{12}?

Correlation of intensity fluctuation of the light

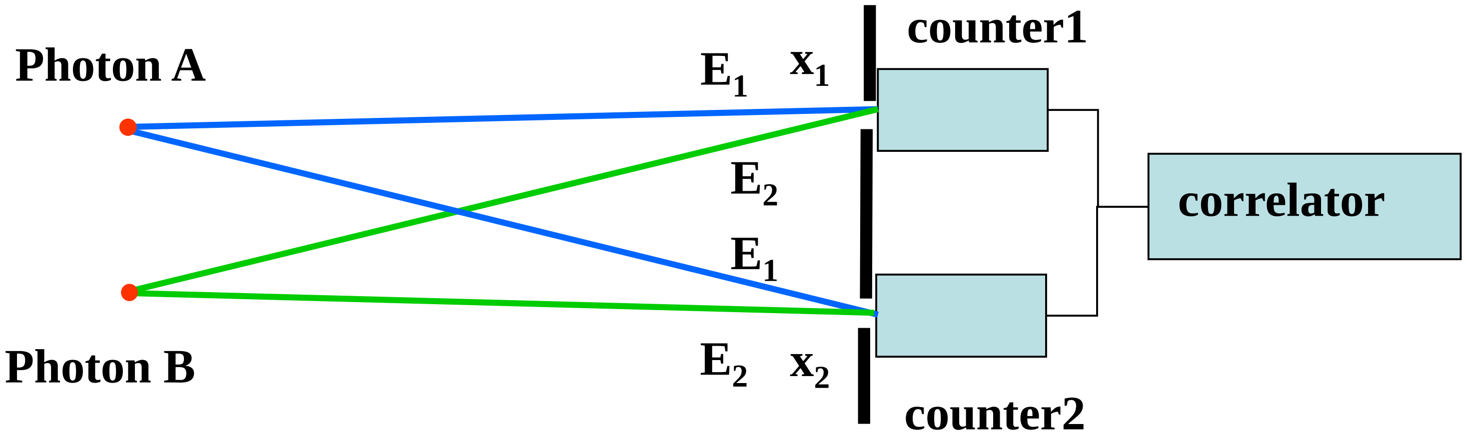


Intensity of light → **number of photons**

Fluctuation of intensity → **photon number fluctuation**

1-1-d. 2ed order spatial coherence of the light

Consider a intensity correlation at different position x_1,x_2 in the time t_1,t_2, respectively.

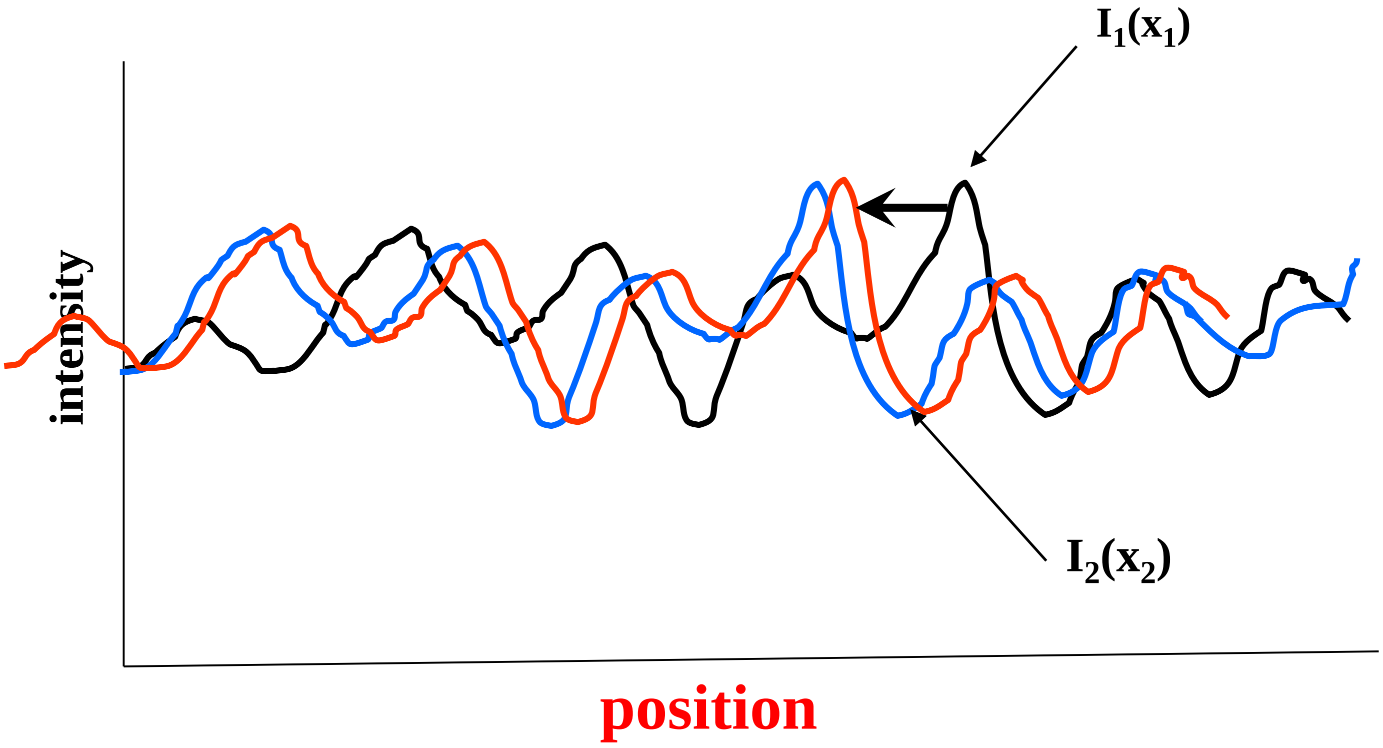


Coincidence count at x_1 and x_2 is given by;

$$P_{12}(x_1, x_2) = \langle E_1(x_1) E_2(x_2) E^*{}_2(x_1) E^*{}_1(x_2) \rangle$$

What is the physical meaning of P_{12}?

Correlation of intensity fluctuation of the light

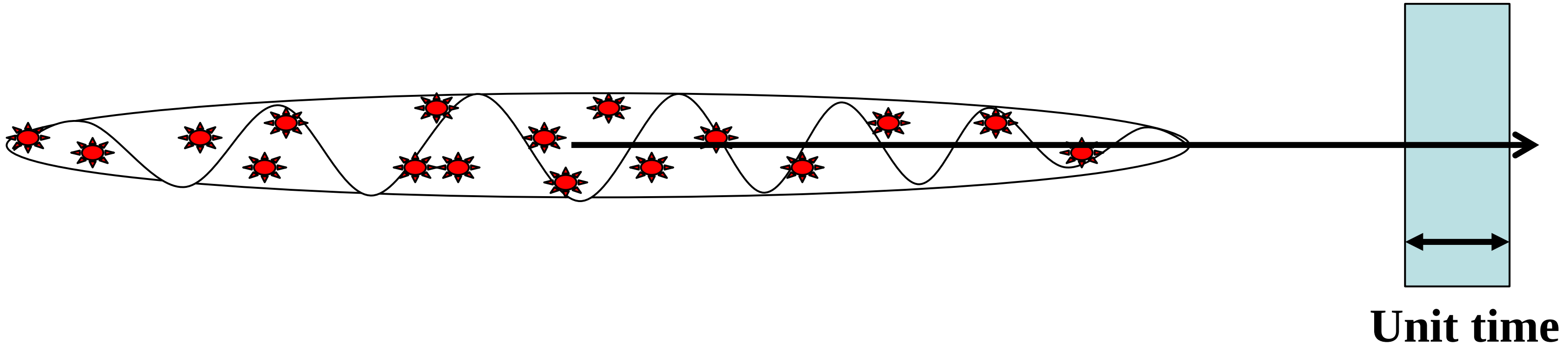


Intensity of light $\Rightarrow$ **number of photons**

Fluctuation of intensity $\Rightarrow$ **photon number fluctuation**

1-2. Photon statistics

Let consider n photons are observed in unit time to the counter in the photon number state $|M\rangle$ (it is m photons are exited in wavepocket).



Fluctuation of photon number is dominated by photon statistics.

1. **n photons are randomly exited in infinite (practically long) length wavepocket.**

   → **n photon coherent state.** **Poisson distribution**

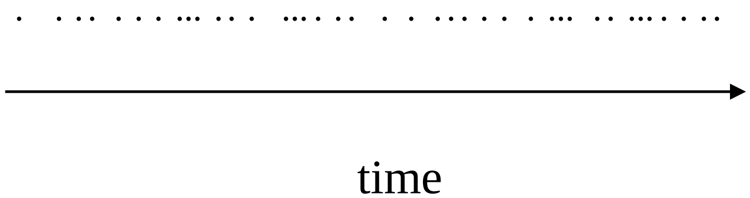
2. **n photons are exited in finite length wavepocket.**

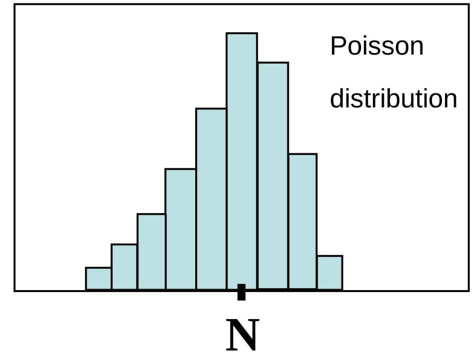
   → **Chaotic state.** **Super-Poisson distribution**

3. **n photons are orderly exited in infinite (practically long) length wavepocket.**

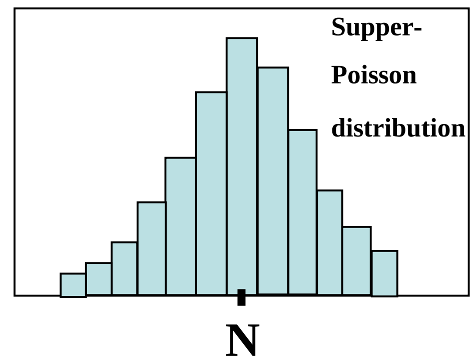
   → **Unbunched state** **Sub-Poisson distribution**

Case 1. coherent state : the provability to find the photon at any time t is almost constant

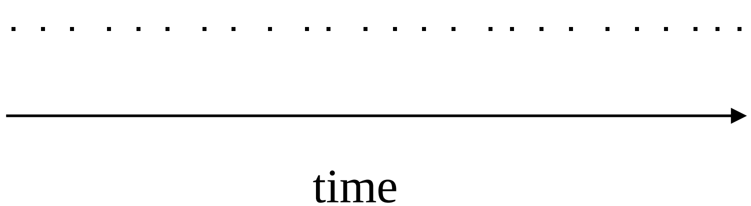


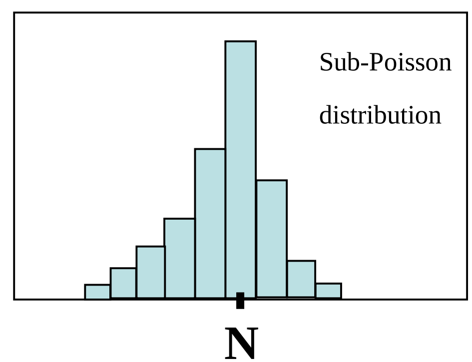


Case 2. Chaotic state: the provability to find the photon at time t is localized in the wavepocket.



Case 3. Unbunched state: the provability to find the photon at t is in more order.





Can we find correct phase operator ?

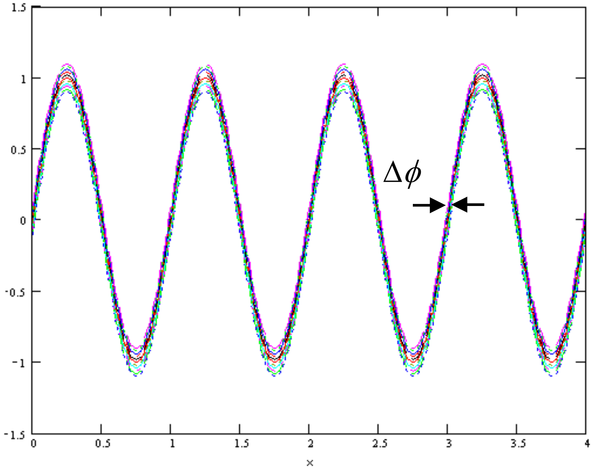
It is still one of remaining great problem in modern optics.

But uncertainty relation

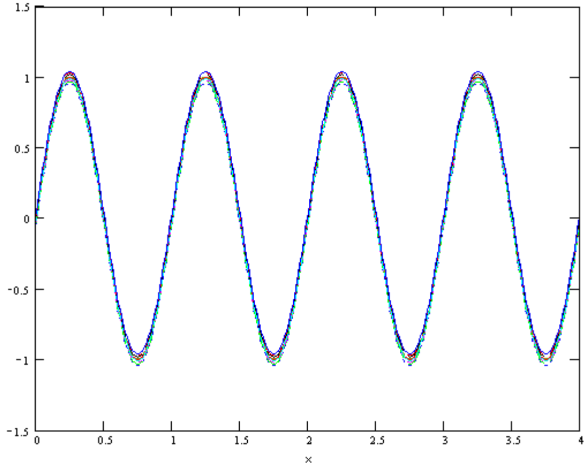
$$\Delta\phi \cdot \Delta n \geq 1/2$$

seems OK for large photon number.

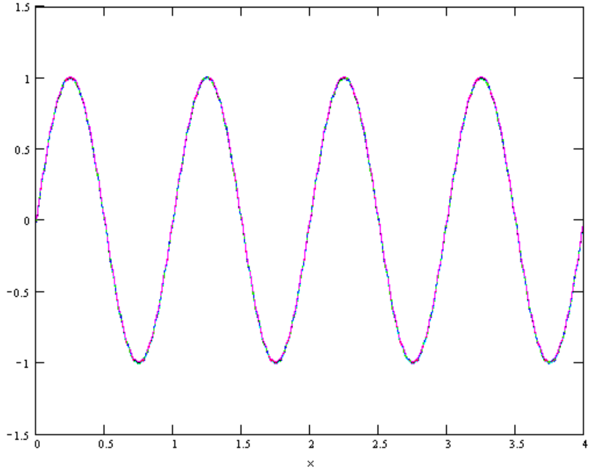
Fluctuation of phase appear in some photon numbers



$|n|^2=50$

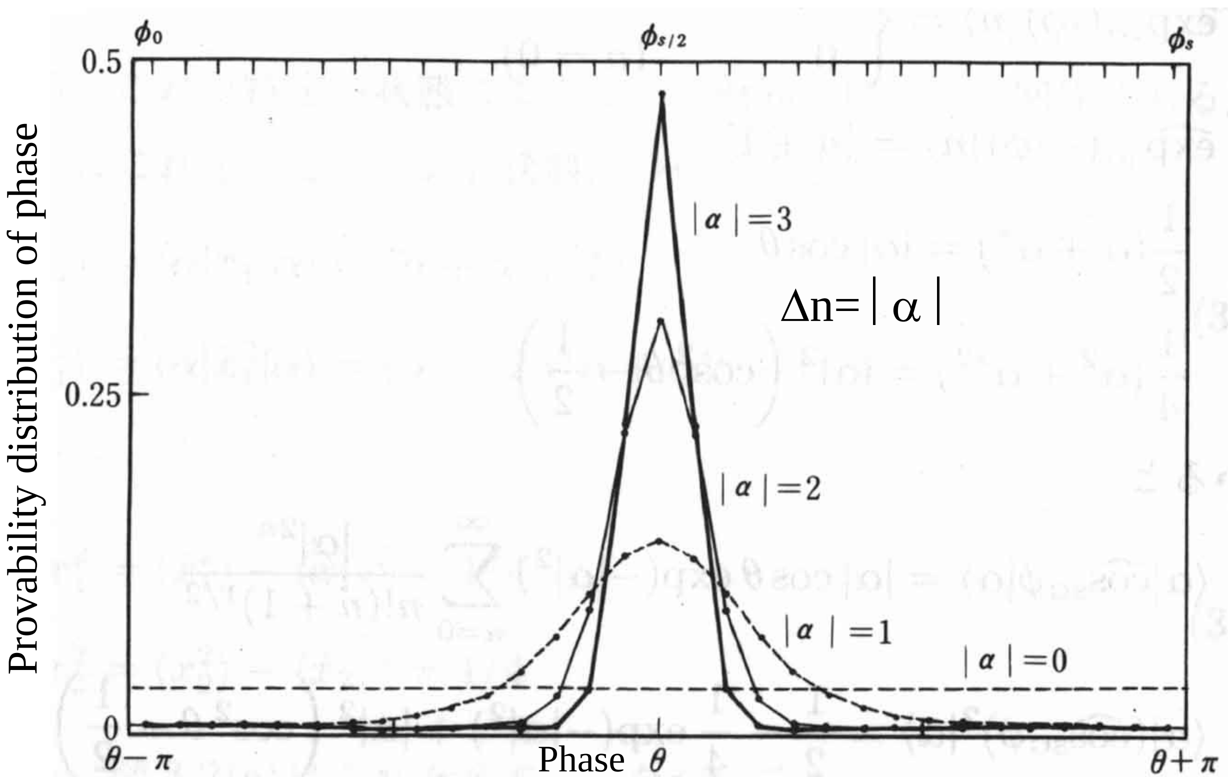


$|n|^2=100$



$|n|^2=500$

Provability distribution of light phase

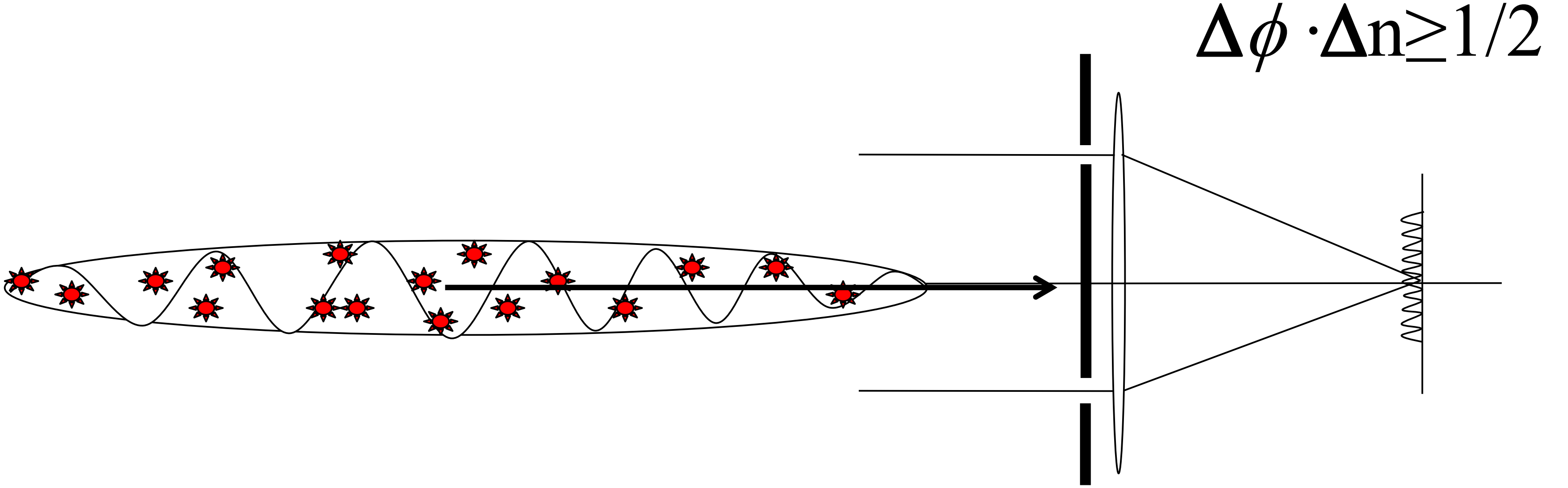


Relation large number n photons exited in single state $|n\rangle$ and n times observation of 1 photon state $|1\rangle$.

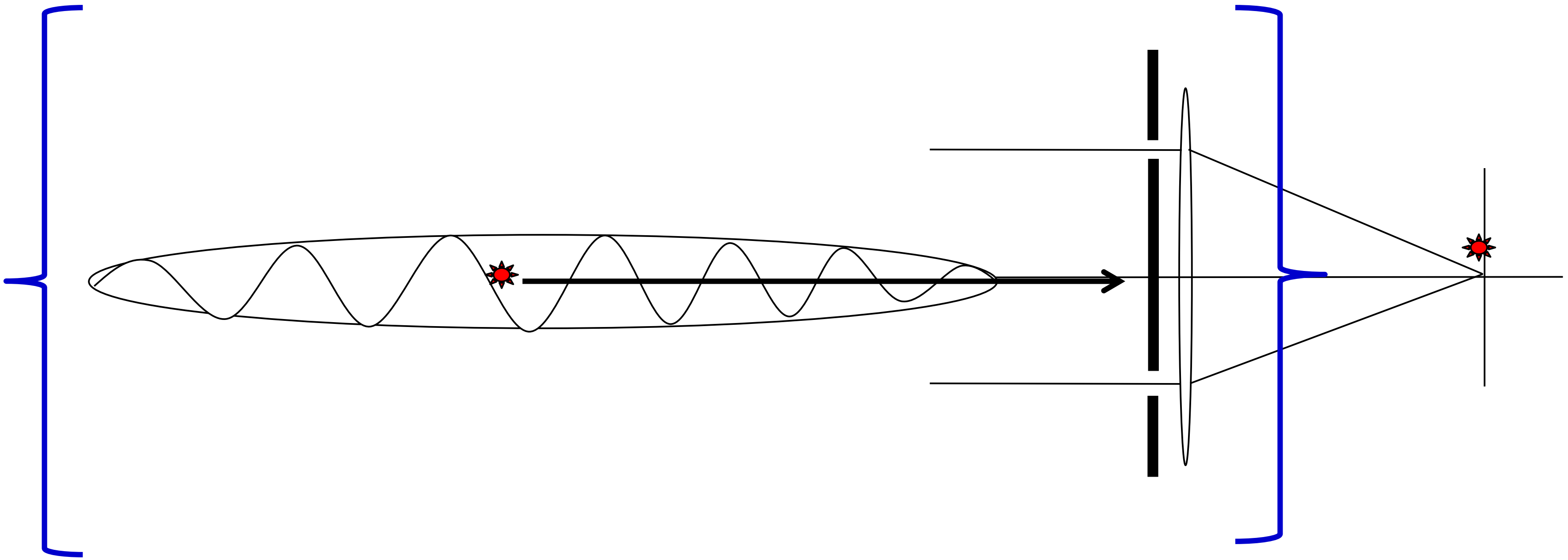
Each 1photon states in independent n times observation are basically do not have 1st-order coherence.

How we can explain independent wavepockets those cams in incoherent manner, but still will make interference fringe in interferometer?

1 times observation of n photons in $|n\rangle$



n times observation of independent $|1\rangle$



More precisely, this problem is same as how to convert n photons in one wavepocket and one photon in n wavepockets.

We assume the most restrictive class of random process of photon statistics to the class of ergodic random process (long temporal average is equal large ensemble average). When n is large,

n times observation of one photon in the wavepocket

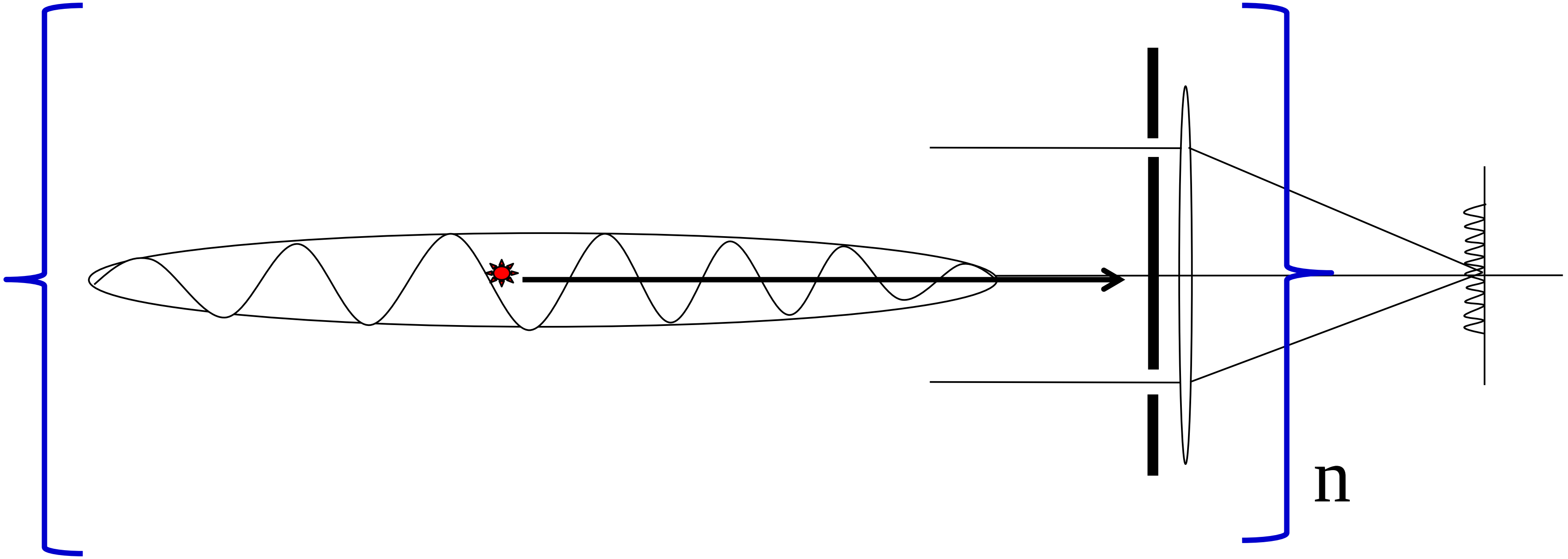
Temporal average

$\Updownarrow$

n photons are exited in the wavepocket.

Ensemble average

n times observation of independent $|1\rangle$



$$\Delta\phi \cdot \Delta n \geq 1/2$$

Lecture 5

Physics of Synchrotron Radiation

The Maxwell equation represents the phenomena of emission of the electromagnetic wave from the charge and current as the sources.

The Green's function recognized as the propagation function of the D' Alembert equation.

The propagation function represents a wave which propagates with light speed emitted from point source.

Let’s start with D’ Alembert equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\boldsymbol{x},t) = -\boldsymbol{f}(\boldsymbol{x},t)$$

Since the signal started at $\boldsymbol{x}'$ and come to $\boldsymbol{x}$ needs time R/c, then corresponding time to the $\boldsymbol{x}'$ is given by.

$$t' = t - \frac{R}{c}$$

R is distance between $\boldsymbol{x}'$ and $\boldsymbol{x}$.

$$R = |\boldsymbol{R}| = |\boldsymbol{x} - \boldsymbol{x}'|$$

This equation gives the solution,

$$\psi(\boldsymbol{x},t)=\frac{1}{4\pi}\int\frac{\boldsymbol{f}\left(\boldsymbol{x'},t-R/c\right)}{R}\,dt'$$

This solution is called Riemann-Lorenz’s lateral potential.

In same way, lateral electromagnetic potential is given by,

$$\phi(\boldsymbol{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x'},t-R/c)}{R}\,dt'$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\mathbf{x'},t-R/c)}{R}\,dt'$$

Using charge density and current density of point charge,

$$\rho(\boldsymbol{x}, t) = q\,\delta(\boldsymbol{x} - \boldsymbol{z}(t))$$

$$\boldsymbol{J}(\boldsymbol{x}, t) = q\,\boldsymbol{v}(t)\,\delta(\boldsymbol{x} - \boldsymbol{z}(t))$$

in here, $\boldsymbol{R}(t) = \boldsymbol{x} - \boldsymbol{z}(t)$ is distance between observation point and point charge at t'.

Replacing ρ and $\boldsymbol{J}$ in lateral electromagnetic potential by these equations,

$$\phi(x\,,\,\mathrm{t}) = \frac{q}{4\pi\varepsilon_0}\int \frac{1}{\mathrm{R}(\mathrm{t'})}\,\delta\left(\mathrm{t} - \mathrm{R}(\mathrm{t'})/\mathrm{c} - \mathrm{t'}\right)\mathrm{dt'}$$

$$A(x\,,\,\mathrm{t}) = \frac{\mu_0 q}{4\pi}\int \frac{v(\mathrm{t'})}{\mathrm{R}(\mathrm{t'})}\,\delta\left(\mathrm{t} - \mathrm{R}(\mathrm{t'})/\mathrm{c} - \mathrm{t'}\right)\mathrm{dt'}$$

Note, using the Doppler factor is given by using,

$$t = t' + \frac{R(t')}{c}$$

$$K(t') = \frac{\partial t(t')}{\partial t'} = \frac{\partial t'}{\partial t'} + \frac{1}{c}\frac{\partial R(t')}{\partial t'} = 1 - \frac{v}{c} \cos\boldsymbol{\theta} = 1 - \boldsymbol{n} \cdot \boldsymbol{\beta}$$

We can derive **Lienard-Wiechert** electromagnetic potential.

$$\phi(\boldsymbol{x}, t') = \frac{q}{4\pi\varepsilon_0} \frac{1}{K(t')R(t')}$$

$$\boldsymbol{A}(\boldsymbol{x}, t') = \frac{q\mu_0}{4\pi} \frac{\boldsymbol{v}(t')}{K(t')R(t')}$$

The electric field and the magnetic field are given by,

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

Replacing ϕ and $\boldsymbol{A}$ by lateral electromagnetic potential, We can derive the electric field and the magnetic field.

$$\phi(x, \mathrm{t}) = \frac{q}{4\pi\varepsilon_0} \int \frac{1}{\mathrm{R}(\mathrm{t}')} \delta\left(\mathrm{t} - \mathrm{R}(\mathrm{t}')/\mathrm{c} - \mathrm{t}'\right) \mathrm{dt}'$$

$$A(x, \mathrm{t}) = \frac{\mu_0 q}{4\pi} \int \frac{v(\mathrm{t}')}{\mathrm{R}(\mathrm{t}')} \delta\left(\mathrm{t} - \mathrm{R}(\mathrm{t}')/\mathrm{c} - \mathrm{t}'\right) \mathrm{dt}'$$

$$= \frac{q}{4\pi c^2 \varepsilon_0} \int \frac{v(\mathrm{t}')}{\mathrm{R}(\mathrm{t}')} \delta\left(\mathrm{t} - \mathrm{R}(\mathrm{t}')/\mathrm{c} - \mathrm{t}'\right) \mathrm{dt}'$$

The result for the electric field is,

$$\boldsymbol{E}(\boldsymbol{x},\mathrm{t})=\frac{\mathrm{q}}{4\boldsymbol{\pi\varepsilon}_0}\int\left\{\frac{\boldsymbol{n}}{\mathrm{R}^2}+\frac{1}{\mathrm{R\,c}}(\boldsymbol{n}-\boldsymbol{\beta})\frac{\partial}{\partial \mathrm{t}}\right\}\boldsymbol{\delta}\left(\mathrm{t}-\frac{\mathrm{R}(\mathrm{t}')}{\mathrm{c}}-\mathrm{t}'\right)\mathrm{dt}'$$

$$=\frac{\mathrm{q}}{4\boldsymbol{\pi\varepsilon}_0}\left\{\frac{\boldsymbol{n}}{\mathrm{K\,R}^2}+\frac{1}{\mathrm{c}}\left(\frac{\boldsymbol{n}-\boldsymbol{\beta}}{\mathrm{K\,R}}\right)\right\}$$

Using Doppler factor,

$$K(t') = \frac{\partial t(t')}{\partial t'}$$

Then,

$$\frac{\partial}{\partial t'} = K(t')\frac{\partial}{\partial t(t')}$$

We got

$$K(t') = 1 + \frac{1}{c}\frac{\partial R(t')}{\partial t} \qquad \frac{1}{K(t')} = \left(1 + \frac{1}{c}\frac{\partial R(t')}{\partial t}\right)^{-1} \cong 1 - \frac{1}{c}\frac{\partial R(t')}{\partial t}$$

$$\beta(t') = -\frac{K(t')}{c}\frac{\partial}{\partial t}\left(R(t')\boldsymbol{n}\right)$$

The electric field becomes,

$$\boldsymbol{E}(\boldsymbol{x},\mathrm{t}) = \frac{\mathrm{q}}{4\pi\varepsilon_0}\left\{\frac{\boldsymbol{n}}{\mathrm{K\,R}^2} + \frac{1}{\mathrm{c}}\left(\frac{\boldsymbol{n}-\boldsymbol{\beta}}{\mathrm{K\,R}}\right)\right\}$$

$$= \frac{\mathrm{q}}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}}{\mathrm{R}^2}\left(1-\frac{1}{\mathrm{c}}\frac{\partial \mathrm{R}(\mathrm{t}')}{\partial \mathrm{t}}\right) + \frac{1}{\mathrm{c}}\frac{\partial}{\partial \mathrm{t}}\left\{\frac{\boldsymbol{n}}{\mathrm{K\,R}} - \frac{\boldsymbol{\beta}}{\mathrm{K\,R}}\right\}\right]$$

$$= \frac{\mathrm{q}}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}}{\mathrm{R}^2}\left(1-\frac{1}{\mathrm{c}}\frac{\partial \mathrm{R}(\mathrm{t}')}{\partial \mathrm{t}}\right) + \frac{1}{\mathrm{c}}\frac{\partial}{\partial \mathrm{t}}\left\{\frac{\boldsymbol{n}}{\mathrm{R}}\left(1-\frac{1}{\mathrm{c}}\frac{\partial \mathrm{R}(\mathrm{t}')}{\partial \mathrm{t}}\right) - \frac{\boldsymbol{\beta}}{\mathrm{K\,R}}\right\}\right]$$

$$= \frac{\mathrm{q}}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}}{\mathrm{R}^2}\left(1-\frac{1}{\mathrm{c}}\frac{\partial \mathrm{R}(\mathrm{t}')}{\partial \mathrm{t}}\right) + \frac{1}{\mathrm{c}}\frac{\partial}{\partial \mathrm{t}}\left\{\frac{\boldsymbol{n}}{\mathrm{R}}\left(1-\frac{1}{\mathrm{c}}\frac{\partial \mathrm{R}(\mathrm{t}')}{\partial \mathrm{t}}\right)\right.\right.$$

$$\left.\left. + \frac{1}{\mathrm{cR}}\frac{\partial}{\partial \mathrm{t}}\left(\mathrm{R}(\mathrm{t}')\boldsymbol{n}\right)\right\}\right]$$

Clean up the equation, the electric field is given by,

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}}{R^2} + \frac{R}{c}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{n}}{R^2}\right) + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{n}}{\partial t^2}\right]$$

Same calculation for the magnetic field,

Remember the vector potential ***A*** is given by,

$$A(x\,,\,\mathrm{t}) = \frac{\mu_0 q}{4\pi}\int \frac{v(\mathrm{t'})}{\mathrm{R}(\mathrm{t'})}\,\delta\big(\mathrm{t} - \mathrm{R}(\mathrm{t'})/\mathrm{c} - \mathrm{t'}\big)\,\mathrm{dt'}$$

And the magnetic field ***B*** is given by

$$B = \nabla \times A$$

Then,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$= \frac{\boldsymbol{\mu}_0 \mathrm{q}}{4\boldsymbol{\pi}} \int \left(-\mathrm{c}\, \frac{\boldsymbol{n} \times \boldsymbol{\beta}}{\mathrm{R}^2} - \frac{\boldsymbol{n} \times \boldsymbol{\beta}}{\mathrm{R}^2} \frac{\partial}{\partial \mathrm{t}} \right) \boldsymbol{\delta}\left(\mathrm{t} - \mathrm{R}\left(\mathrm{t}'\right)/\mathrm{c} - \mathrm{t}'\right) \mathrm{dt}'$$

$$= \frac{\boldsymbol{\mu}_0 \mathrm{q}}{4\boldsymbol{\pi}} \left[-\mathrm{c}\, \frac{\boldsymbol{n} \times \boldsymbol{\beta}}{\mathrm{R}^2} - \frac{\partial}{\partial \mathrm{t}} \left(\frac{\boldsymbol{n} \times \boldsymbol{\beta}}{\mathrm{K}\,\mathrm{R}} \right) \right]$$

$$\boldsymbol{\beta}(\mathrm{t}') = \frac{\boldsymbol{v}(\mathrm{t}')}{\boldsymbol{c}}$$

$$= \frac{1}{\boldsymbol{c}} \frac{\partial}{\partial \mathrm{t}'} \left(\mathrm{R}(\mathrm{t}')\boldsymbol{n}\right) = \frac{1}{\boldsymbol{c}} \frac{\partial \mathrm{t}(\mathrm{t}')}{\partial \mathrm{t}'} \frac{\partial}{\partial \mathrm{t}(\mathrm{t}')} \left(\mathrm{R}(\mathrm{t}')\boldsymbol{n}\right)$$

Then using

$$\boldsymbol{\beta}(\mathrm{t}') = -\frac{\mathrm{K}(\mathrm{t}')}{\mathrm{c}} \frac{\partial}{\partial \mathrm{t}} \left(\mathrm{R}(\mathrm{t}')\boldsymbol{n}\right)$$

and also using

$$\boldsymbol{n} \times \boldsymbol{\beta} = -\frac{1}{\mathrm{c}} \mathrm{K}\ \mathrm{R}\ \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\partial \mathrm{t}}$$

Then the magnetic field becomes,

$$\boldsymbol{B} = \frac{\mu_0 q}{4\pi}\left[\frac{1}{R}\boldsymbol{n}\times\frac{\partial \boldsymbol{n}}{\partial t} + \frac{1}{c}\boldsymbol{n}\times\frac{\partial^2 \boldsymbol{n}}{\partial t^2}\right]$$

$$= \frac{1}{c}\boldsymbol{n}\times\boldsymbol{E}$$

Convert into standard from using β=v/c,

$$\mathbf{E}(\mathbf{x},t)=e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\mathrm{ret}}+\frac{e}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{\mathrm{ret}}$$

$$=e\left[\frac{\mathbf{n}}{R^2}+\frac{R}{kc}\frac{d}{dt'}\frac{\mathbf{n}}{R^2}\right]_{\mathrm{ret}}+\frac{e}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{\mathrm{ret}}$$

Assuming the field is observed in far, we can retain , then electric field becomes ;

$$\mathbf{E}(\mathbf{x},t)=\frac{e}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{\mathrm{ret}}$$

$$\mathbf{B}(\mathbf{x},t)=\frac{\mathbf{n}\times\mathbf{E}(\mathbf{x},t)}{c}$$

Convert into standard from using $\beta = v/c$,

$$\mathbf{E}(\mathbf{x},t) = e\boxed{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]} + \boxed{\frac{e}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]}$$

Electric field proportional to acceleration

Electric field proportional to velocity

$$= e\left[\frac{\mathbf{n}}{R^2} + \frac{R}{k\,c}\frac{d}{dt'}\frac{\mathbf{n}}{R^2}\right] + \frac{e}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]$$

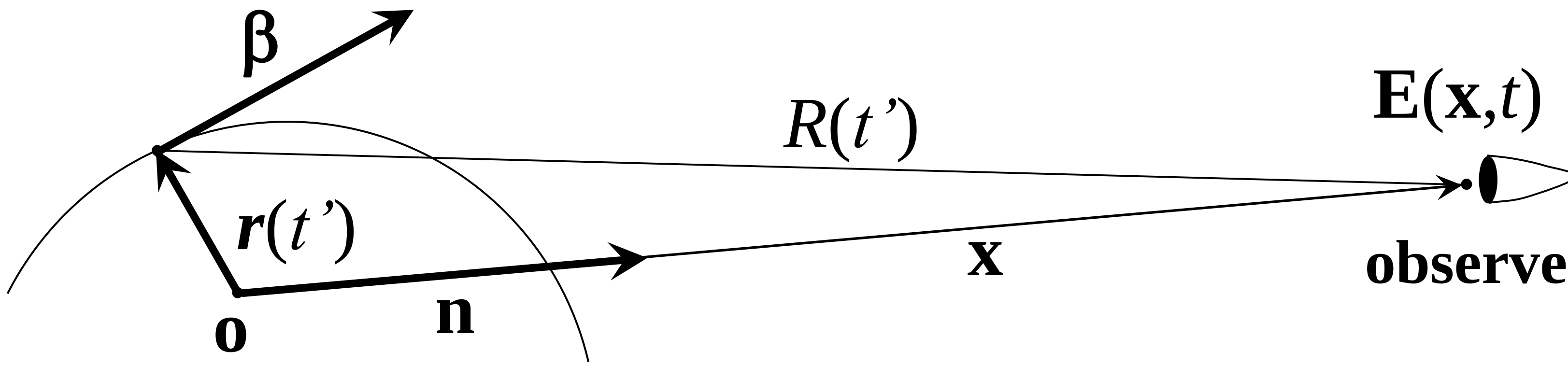
Assuming the field is observed in far, we can retain , then electric field becomes ;

$$\mathbf{E}(\mathbf{x},t) = \frac{e}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]$$

$$\mathbf{B}(\mathbf{x},t) = \frac{\mathbf{n}\times\mathbf{E}(\mathbf{x},t)}{c}$$

Radiation from accelerated charge in circular trajectory

t :observer time



t' : moving frame time

Doppler factor between t' and t

$$k = \frac{dt}{dt'} = 1 - \mathbf{n}\cdot\boldsymbol{\beta} = 1 - \beta\cos(\theta)$$

Introducing the Vector potential ;

$$\mathbf{A}(\mathbf{t}) = \left(\frac{\mathbf{c}}{\mathbf{4}\pi}\right)^{1/2} [\mathbf{R}\ \mathbf{E}]_{\mathrm{ret}}$$

Jackson p673

$$= \left(\frac{\mathbf{e}^2}{\mathbf{4}\pi\mathbf{c}}\right)^{1/2} \left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\mathrm{ret}}$$

Fourier Transfer of A(t **) gives the vector potential in frequency domain A(** ω **),**

$$\mathbf{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(t) e^{i\omega t}\ dt$$

$$\mathbf{A}(\omega) = \left(\frac{\mathbf{e}^2}{8\pi^2\mathbf{c}}\right)^{1/2} \int_{-\infty}^{\infty} \mathbf{e}^{\mathbf{i}\omega\mathbf{t}} \left[\frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\mathbf{ret}} \mathbf{dt}$$

Jackson p675

$$t = t' + \frac{R(t')}{c} \quad then, \quad dt = dt'(1 - \boldsymbol{\beta}\cdot\mathbf{n}) \quad \text{Doppler factor}$$

$$\mathbf{A}(\omega) = \left(\frac{\mathbf{e}^2}{8\pi^2\mathbf{c}}\right)^{1/2} \int_{-\infty}^{\infty} \mathbf{e}^{\mathbf{i}\omega\left(\mathbf{t}' + \frac{\mathrm{R}(\mathrm{t}')}{\mathrm{c}}\right)} \frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\,\mathbf{dt}'$$

$$\text{Approximate} \quad R(t') \approx x - \mathbf{n}\cdot\mathbf{r}(t') \quad \text{Then,} \quad \frac{R(t')}{c} \approx -\mathbf{n}\cdot\mathbf{r}(t')$$

Jackson p675

$$\mathbf{A}(\omega) = \left(\frac{\mathbf{e}^2}{8\pi^2\mathbf{c}}\right)^{1/2} \int_{-\infty}^{\infty} \mathbf{e}^{\mathbf{i}\omega\left(\mathbf{t}' - \frac{\mathbf{n}\cdot\mathbf{r}(\mathbf{t}')}{\mathrm{c}}\right)} \frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\,\mathbf{dt}'$$

$$\frac{\mathbf{d}}{\mathbf{dt}'}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right] = \frac{\mathbf{n}\times[\mathbf{n}-\boldsymbol{\beta}]\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}$$

$$\mathbf{A}(\omega)=\left(\frac{e^2}{8\pi^2 c}\right)^{1/2}\int_{-\infty}^{\infty} e^{i\omega\left(t'-\frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)}\frac{d}{dt'}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]dt'$$

Jackson p675

Then an integration by parts,

$$\mathbf{A}(\omega)=\left(\frac{e^2}{8\pi^2 c}\right)^{1/2}\int_{-\infty}^{\infty} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\, e^{i\omega\left(t'-\frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)}\,dt'$$

Jackson p676

Still difficult to understand the physical image .

How we can understand contribution from ±infinity time in integral?

Return to the Doppler factor

$$k = \frac{\partial t(t')}{\partial t'} = 1 - \beta \cos\theta = 1 - \boldsymbol{\beta} \cdot \boldsymbol{n}$$

For large θ ;

$$\beta = |\, \boldsymbol{\beta} \,| \qquad\qquad \beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$1 - \beta = 1 - \sqrt{1 - \frac{1}{\gamma^2}} \approx \frac{1}{2\gamma^2} << 1$$

$$k \approx 0(1)$$

For small θ ;

$$k = \frac{dt}{dt'} = 1 - \sqrt{1 - \frac{1}{\gamma^2}}\cos\theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$ **Far field approximation**

$$k = \frac{dt}{dt'} = 1 - \sqrt{1 - \frac{1}{\gamma^2}}\left(1 - \frac{\theta^2}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\gamma^2} + \theta^2\right)$$

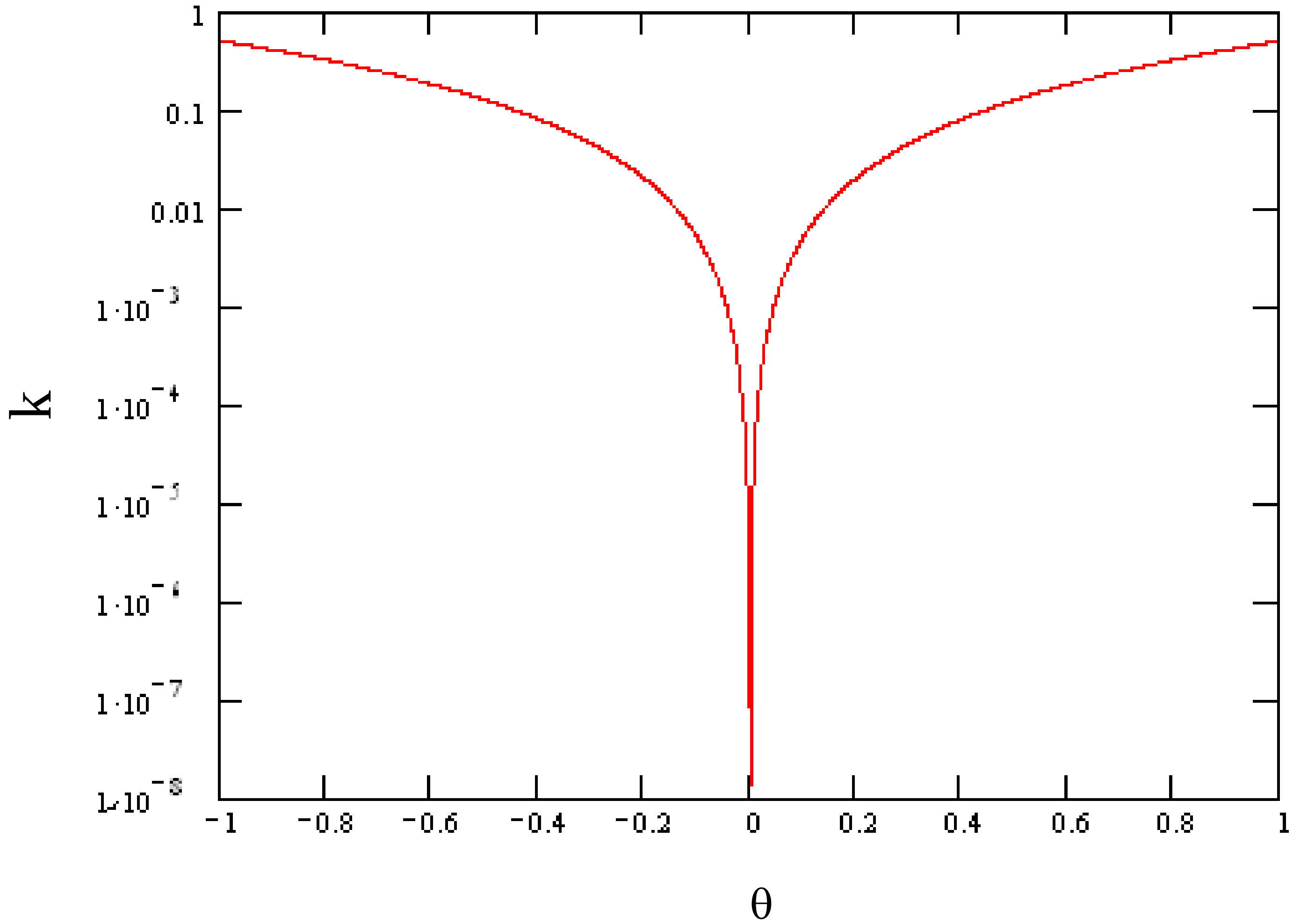
Remember Tayler expansion of cosine

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots\ldots\ldots\ldots$$

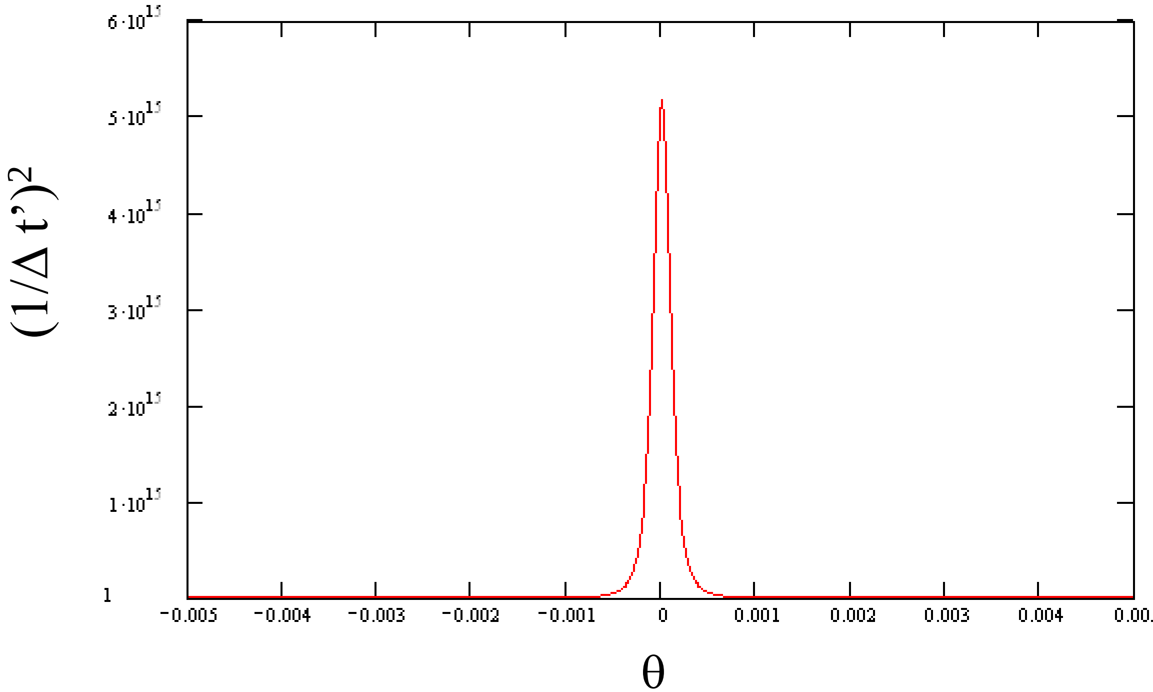
So, for small θ, neglecting higher than

4th power term

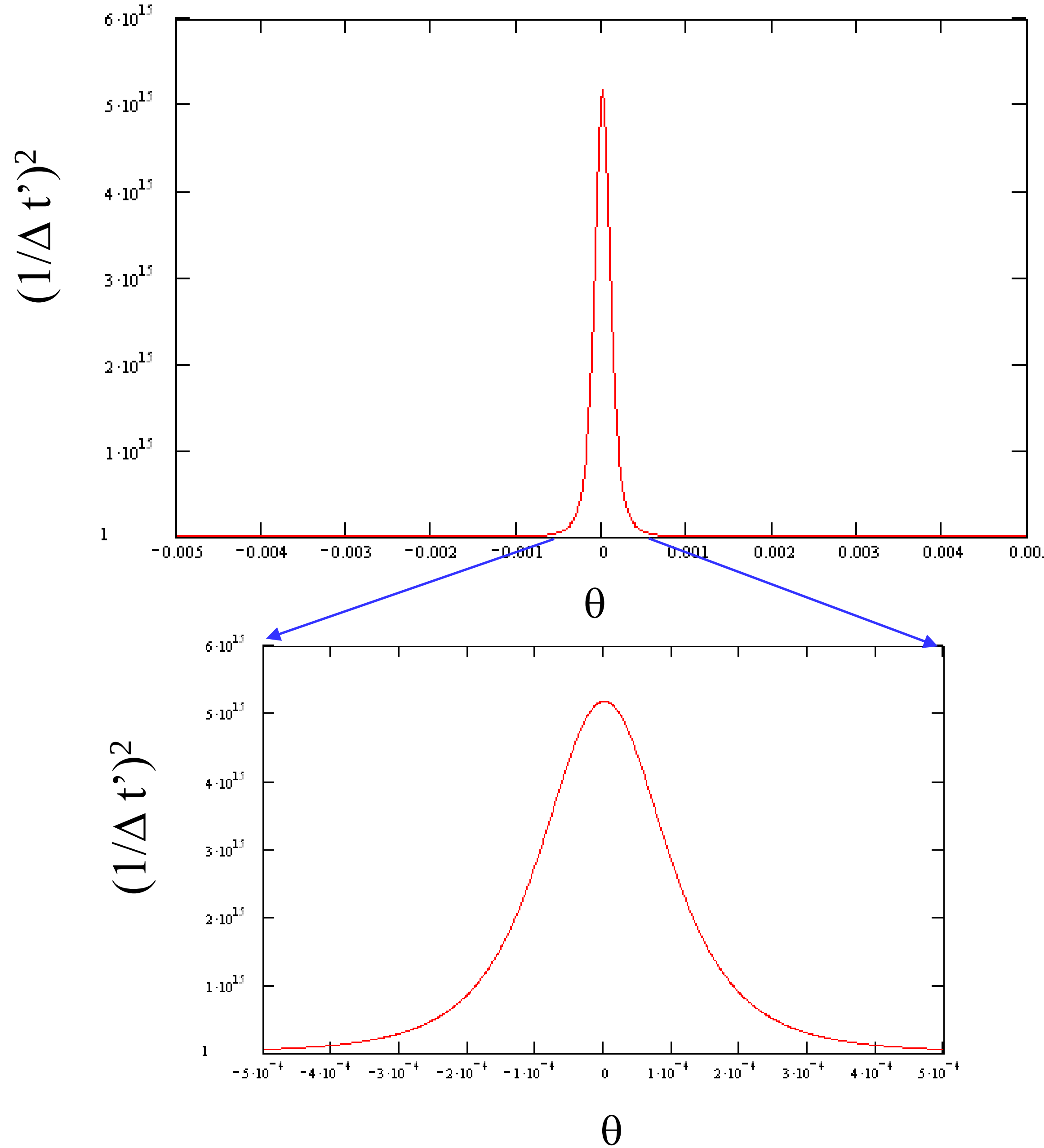
$$\cos\theta = 1 - \frac{\theta^2}{2!}$$



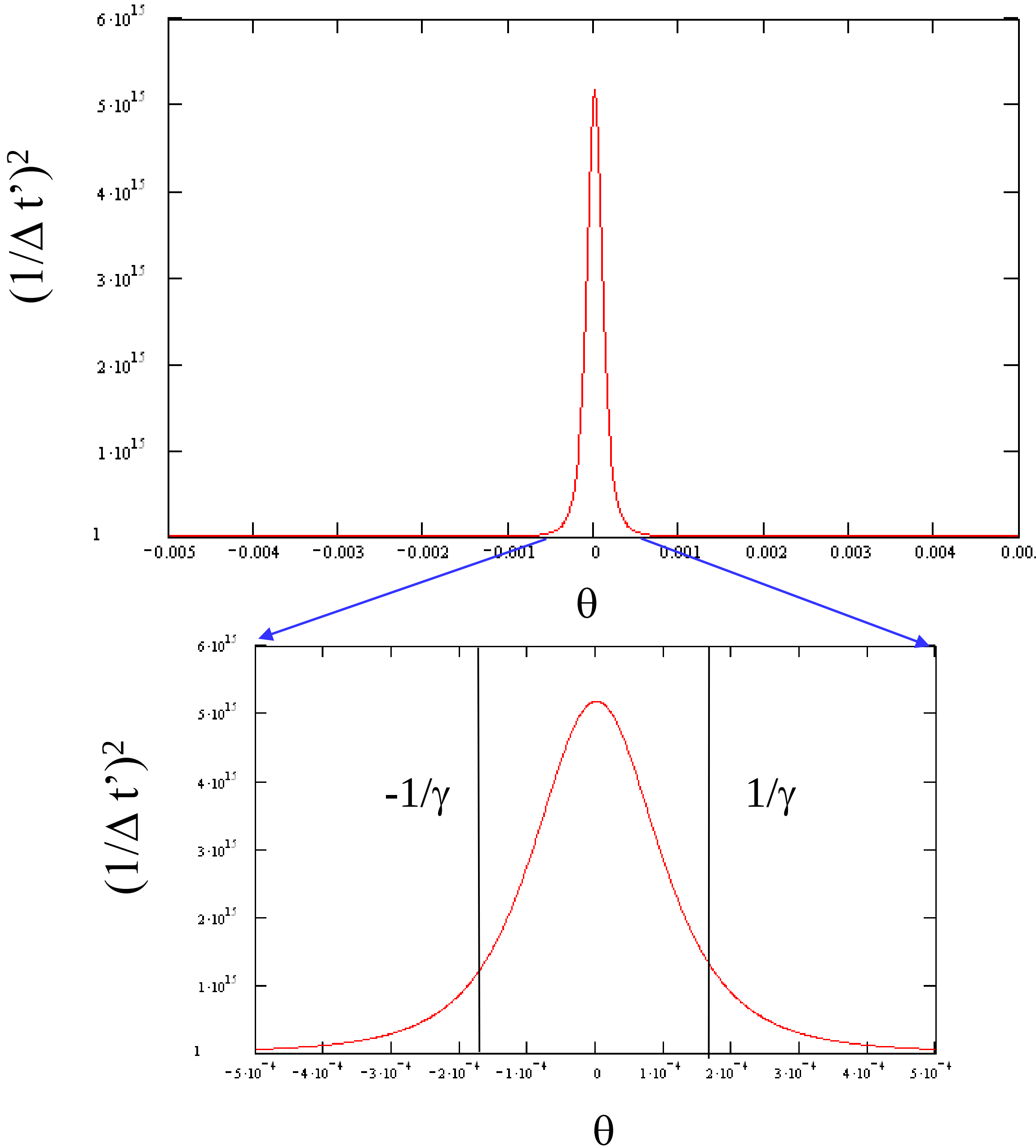
1
0.1
0.01
1·10^-3
1·10^-4
1·10^-5
1·10^-6
1·10^-7
1·10^-8
k
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
θ



$(1/\Delta t')^2$
$6 \cdot 10^{15}$
$5 \cdot 10^{15}$
$4 \cdot 10^{15}$
$3 \cdot 10^{15}$
$2 \cdot 10^{15}$
$1 \cdot 10^{15}$
1
−0.005
−0.004
−0.003
−0.002
−0.001
0
0.001
0.002
0.003
0.004
0.005
θ

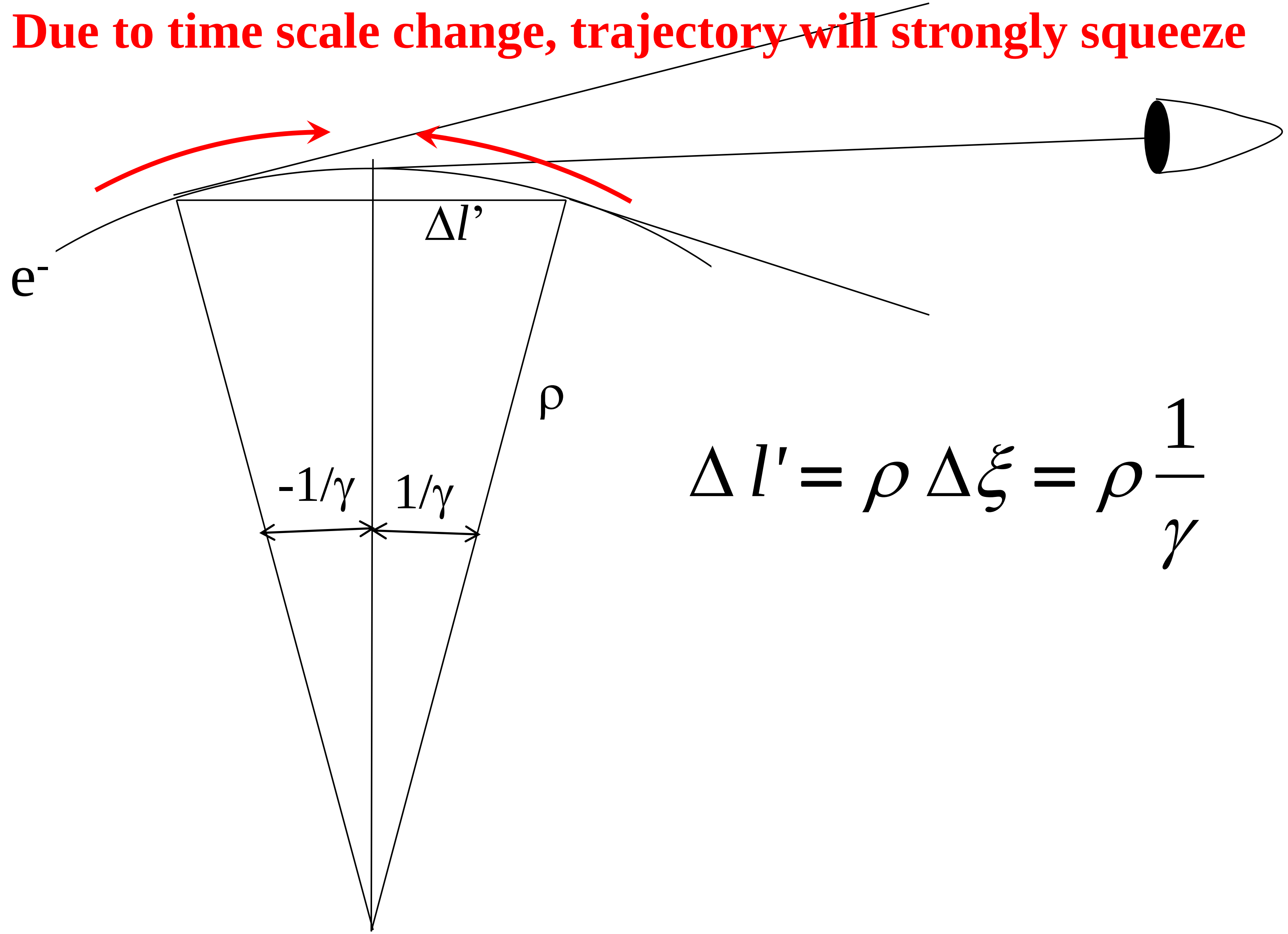


$(1/\Delta t')^2$
$6\cdot10^{15}$
$5\cdot10^{15}$
$4\cdot10^{15}$
$3\cdot10^{15}$
$2\cdot10^{15}$
$1\cdot10^{15}$
1
−0.005
−0.004
−0.003
−0.002
−0.001
0
0.001
0.002
0.003
0.004
0.005
θ
$(1/\Delta t')^2$
$6\cdot10^{15}$
$5\cdot10^{15}$
$4\cdot10^{15}$
$3\cdot10^{15}$
$2\cdot10^{15}$
$1\cdot10^{15}$
1
$-5\cdot10^{-4}$
$-4\cdot10^{-4}$
$-3\cdot10^{-4}$
$-2\cdot10^{-4}$
$-1\cdot10^{-4}$
0
$1\cdot10^{-4}$
$2\cdot10^{-4}$
$3\cdot10^{-4}$
$4\cdot10^{-4}$
$5\cdot10^{-4}$
θ



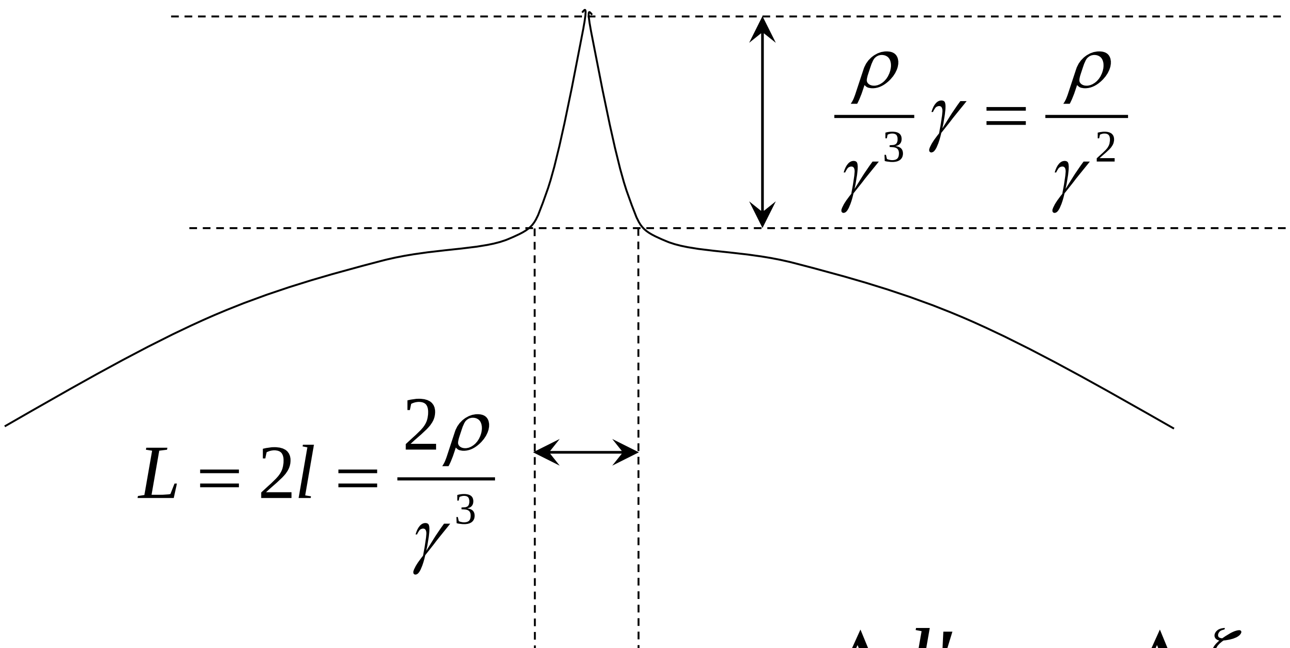
$(1/\Delta t')^2$
θ
$-1/\gamma$
$1/\gamma$

Due to time scale change, trajectory will strongly squeeze



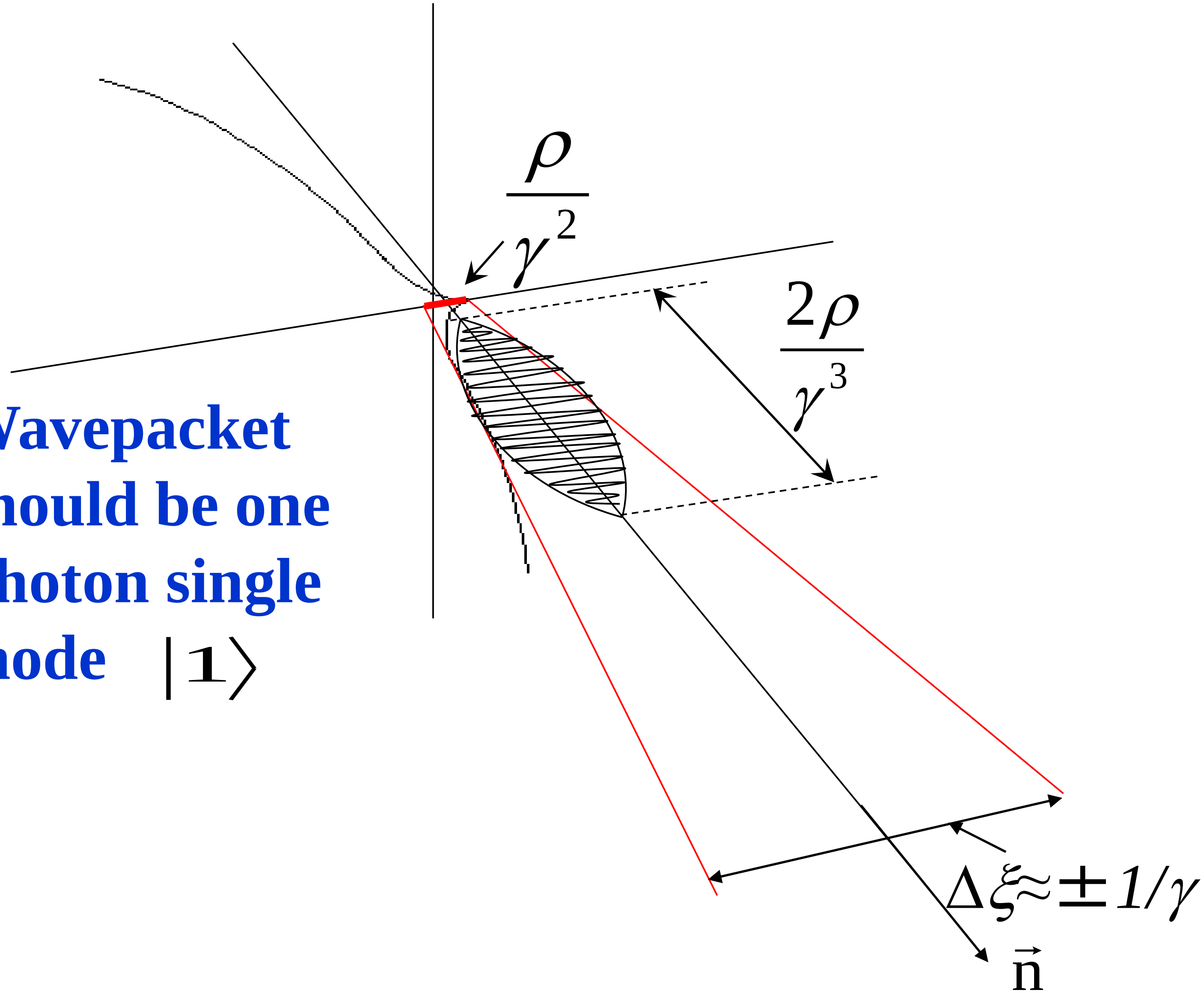
$$\Delta l' = \rho\, \Delta \xi = \rho \frac{1}{\gamma}$$

Then, apparent beam trajectory looks,



$$\Delta l' = \rho\,\Delta\xi = \rho\frac{1}{\gamma}$$

$$\Delta l = \Delta l'\,k = \rho\frac{1}{\gamma^3}$$



Wavepacket should be one photon single mode $|1\rangle$

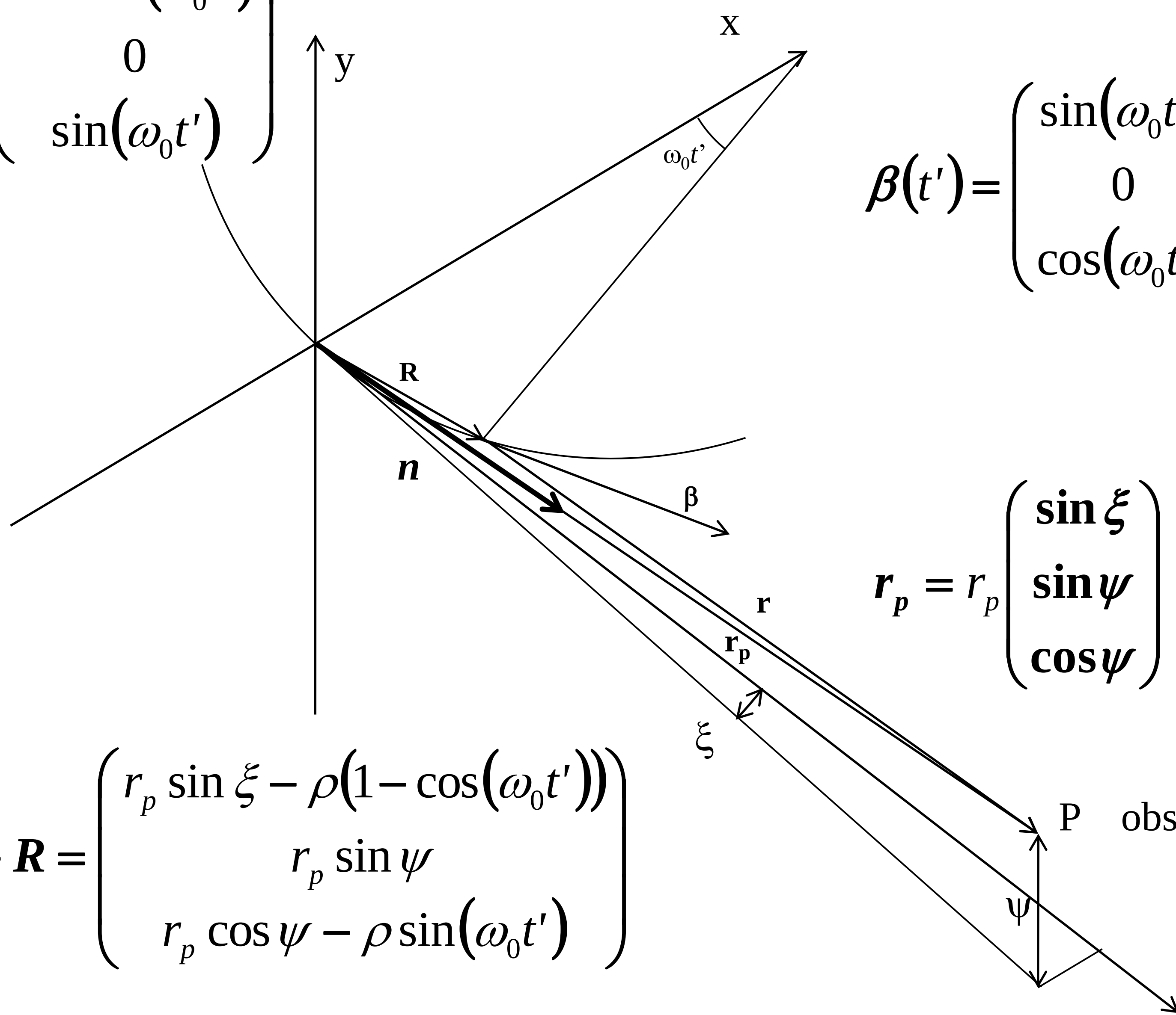
Return to vector potential again

$$\frac{d^2 I}{d\omega d\Omega} = 2\left|\mathbf{A}(\omega)\right|^2$$

Then,

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\; e^{i\omega\left(t' + \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c}\right)} \; dt'\right|^2$$

$$\boldsymbol{R}(t') = \rho\begin{pmatrix} 1-\cos(\omega_0 t') \\ 0 \\ \sin(\omega_0 t') \end{pmatrix}$$



$$\boldsymbol{\beta}(t') = \begin{pmatrix} \sin(\omega_0 t') \\ 0 \\ \cos(\omega_0 t') \end{pmatrix}$$

$$\boldsymbol{r_p} = r_p\begin{pmatrix} \sin\xi \\ \sin\psi \\ \cos\psi \end{pmatrix}$$

$$\boldsymbol{r} = \boldsymbol{r_p} - \boldsymbol{R} = \begin{pmatrix} r_p \sin\xi - \rho(1-\cos(\omega_0 t')) \\ r_p \sin\psi \\ r_p \cos\psi - \rho\sin(\omega_0 t') \end{pmatrix}$$

θ is angle between $\boldsymbol{\beta}$ and $\boldsymbol{n}$

Assuming,

$$\cos(\omega_0 t') \approx 1$$
$$\sin(\omega_0 t') \approx 0$$

$$\boldsymbol{n} = \frac{\boldsymbol{r}}{r_p} = \begin{pmatrix} \sin\xi \\ \sin\psi \\ \cos\psi \end{pmatrix}$$

The vector part of the integral can be written by,

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})=(\boldsymbol{n}\cdot\boldsymbol{\beta})\boldsymbol{n}-\boldsymbol{\beta}$$

$$=\beta\left(\sin\xi\,\sin(\omega_0 t')+\cos\psi\,\cos(\omega_0 t')\right)\begin{pmatrix}\sin\xi\\ \sin\psi\\ \cos\psi\end{pmatrix}-\beta\begin{pmatrix}\sin(\omega_0 t')\\ 0\\ \cos(\omega_0 t')\end{pmatrix}$$

$$=\beta\begin{pmatrix}-\sin(\omega_0 t')\cos^2\xi+\cos\psi\,\cos(\omega_0 t')\sin\xi\\ \sin\psi\,\sin\xi\,\sin(\omega_0 t')+\sin\psi\,\cos\psi\,\cos(\omega_0 t')\\ \sin\xi\,\cos\psi\,\sin(\omega_0 t')-\sin^2\psi\,\cos(\omega_0 t')\end{pmatrix}$$

In here, we can approximate followings ,

$$\sin\left(\omega_0 t'\right) \approx \omega_0 t' \ , \quad \cos\left(\omega_0 t'\right) \approx 1$$

$$\sin\psi \approx \psi \ , \quad \sin\xi \approx \xi \ , \quad \cos\psi \approx 1$$

$$\sin^2\psi \approx 0 \, . \quad \cos^2\psi \approx 1$$

Then, the vector part of the integral

$$\boldsymbol{n} \times \left(\boldsymbol{n} \times \boldsymbol{\beta}\right) = \beta \begin{pmatrix} -\omega_0 t' + \xi \\ \psi \\ 0 \end{pmatrix}$$

Concerning to the exponent ,

Time scale factor *k*(*t'*) is given by ,

$$k(t') = \frac{dt(t')}{dt'} = \frac{1}{2}\left(\frac{1}{\gamma^2} + \theta^2\right)$$

in here,

$$\theta^2 = \psi^2 + (\omega_0 t' + \xi)^2$$

then,

$$k(t') = \frac{dt(t')}{dt'} = \frac{1}{2}\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 + (\omega_0 t')^2 + 2\xi\omega_0 t'\right)$$

$$k(t') = \frac{\boldsymbol{dt(t')}}{\boldsymbol{dt'}} = \frac{1}{2}\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 + (\omega_0 t')^2 + 2\xi\omega_0 t' \right)$$

then,

$$\boldsymbol{dt(t')} = \frac{1}{2}\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 + (\omega_0 t')^2 + 2\xi\omega_0 t' \right)\boldsymbol{dt'}$$

integrating this equation ,

Integrate time scale factor,

$$t(t') = \frac{1}{2}\left(\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2\right)t' + \frac{1}{3}\omega_0{}^2 t'^3 + \xi\omega_0 t'^2\right)$$

in here,

$$\omega_0 \cong \frac{c}{\rho}$$

then,

$$\boldsymbol{t(t')} = \frac{1}{2}\left(\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2\right)t' + \frac{1}{3}\frac{c}{\rho^2}{}^{2}\, t'^3 + \xi\frac{c}{\rho}t'^2\right)$$

Then neglecting quadratic phase term,

$$\omega t(t') \approx \frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t'+\frac{1}{3}\frac{c}{\rho^2}^{2}t'^3\right)$$

then, for ξ=0,

$$\omega t(t') \approx \frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2\right)t'+\frac{1}{3}\frac{c}{\rho^2}^{2}t'^3\right)$$

for ψ=0,

$$\omega t(t') \approx \frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\xi^2\right)t'+\frac{1}{3}\frac{c}{\rho^2}^{2}t'^3\right)$$

Then, the vector potential of parallel component is given by,

$$\mathbf{A}_{\parallel}(\omega) \cong \frac{c}{\rho}\int_{-\infty}^{\infty} \left(t' + \frac{\rho}{c}\xi\right) \mathbf{exp}\,\xi\left\{i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2\right)t' + \frac{c^2}{3\rho^2}t'^3 + \frac{c}{\rho}\xi\, t'^2\right]\right\}dt'$$

in here,

$$\frac{\rho}{c}\xi = \frac{\xi}{\omega_0} \approx 0$$

then,

$$\mathbf{A}_{\parallel}(\omega) \cong \frac{c}{\rho}\int_{-\infty}^{\infty} t'\,\mathbf{exp}\left\{i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2\right)t' + \frac{c^2}{3\rho^2}t'^3 + \frac{c}{\rho}\xi\, t'^2\right]\right\}dt'$$

The vector potential of perpendicular component is given by,

$$\mathbf{A}_{\perp}(\omega)=\psi\int_{-\infty}^{\infty}\mathbf{exp}\left\{i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^{2}}+\psi^{2}+\xi^{2}\right)t'+\frac{c^{2}}{3\rho^{2}}t'^{3}+\frac{c}{\rho}\xi\,t'^{2}\right]\right\}dt'$$

The spectrum intensity distribution I is given by,

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \mathbf{A}_{\parallel}(\omega) + \mathbf{A}_{\perp}(\omega) \right|^2$$

1. Special case as ξ=0 (Schwinger's theory)

Let us consider a special case ξ=0 for horizontal observation angle is zero.

Putting ξ=0, the vector potentials becomes,

$$\mathbf{A}_{II}(\omega) \cong \frac{c}{\rho}\int_{-\infty}^{\infty} t' \exp\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2\right)t' + \frac{c^2}{3\rho^2}t'^3\right]\right\}dt'$$

$$\mathbf{A}_{\perp}(\omega) = \psi\int_{-\infty}^{\infty} \exp\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2\right)t' + \frac{c^2}{3\rho^2}t'^3\right]\right\}dt'$$

A change of variable to

$$x = \frac{ct'}{\rho\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}}}$$

them,

$$t' = \frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}} x$$

$$dt' = \frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}} dx$$

$$\zeta = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{3}{2}}$$

the integrals in $A_{II}(\omega)$ into;

$$\mathbf{A}_{II}(\omega)=\frac{c}{\rho}\cdot\left(\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}}\right)\int_{-\infty}^{\infty}x\exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}}\frac{\rho}{c}x\right)\right.$$

$$\left.+\frac{\mathbf{c^2}}{\mathbf{3}\rho^{\mathbf{2}}}\frac{\rho^3}{c^3}\left(\frac{1}{\gamma^2}+\psi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}}\right]\cdot\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}}dx$$

$$=\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)\int_{-\infty}^{\infty}x\exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{3}{2}}\frac{\rho}{c}x\right)+\frac{\rho}{\mathbf{3}c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{3}{2}}\right]dx$$

the integrals in $A_\perp(\omega)$ into;

$$\mathbf{A}_\perp(\omega) = \psi \cdot \int_{-\infty}^{\infty} x \exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}} \frac{\rho}{c} x \right.\right.$$

$$\left.\left. + \frac{\mathbf{c^2}}{\mathbf{3}\rho^{\mathbf{2}}}\frac{\rho^3}{c^3}\left(\frac{1}{\gamma^2}+\psi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}} \right]\cdot\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}} dx$$

$$= \psi\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{\mathbf{1}}{\mathbf{2}}} \int_{-\infty}^{\infty} x \exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{3}{2}} \frac{\rho}{c} x \right) + \frac{\rho}{\mathbf{3}c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{3}{2}} \right] dx$$

Introducing following parameter ζ ;

$$\zeta = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{3}{2}}$$

Then, we can transform the integrals in $\mathbf{A}_{II}(\omega)$ and $\mathbf{A}_{\perp}(\omega)$ into the form;

$$\mathbf{A}_{II}(\omega) = \frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)\int_{-\infty}^{\infty} x \ \exp\left\{\mathbf{i}\,\frac{\mathbf{3}}{\mathbf{2}}\zeta\left[\ x+\frac{\mathbf{1}}{3}x^{\mathbf{3}}\right]\right\}\mathbf{d}x$$

$$\mathbf{A}_{\perp}(\omega) = \psi\,\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} \exp\left\{\mathbf{i}\,\frac{\mathbf{3}}{\mathbf{2}}\zeta\left[\ x+\frac{\mathbf{1}}{3}x^{\mathbf{3}}\right]\right\}\mathbf{d}x$$

These integrals are identifiable as Airy integrals, and alternatively as modified Bessel functions;

$$\int_0^\infty x\, sin\left(\frac{3}{2}\zeta\left(x+\frac{1}{3}x^3\right)\right)dx = \frac{1}{\sqrt{3}}K_{\frac{2}{3}}(\zeta)$$

$$\int_0^\infty cos\left(\frac{3}{2}\zeta\left(x+\frac{1}{3}x^3\right)\right)dx = \frac{1}{\sqrt{3}}K_{\frac{1}{3}}(\zeta)$$

Consequently the intensity radiated per unit frequency interval per unit solid angle is given by;

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2 \left(\frac{1}{\gamma^2} + \psi^2\right)^2 \left[K_{2/3}^2(\zeta) + \frac{\psi^2}{\frac{1}{\gamma^2} + \psi^2} K_{1/3}^2(\zeta) \right]$$

2. General case, $\xi \neq 0$ T. Mitsuhashi 2015

In more general case, for instantaneous horizontal distribution, we need not neglecting the horizontal observation angle ξ.

In this case, the mathematical treatment is bit complicated.

Let start again, vector potentials,

$$\mathbf{A}_{\parallel}(\omega) \cong \frac{\boldsymbol{c}}{\rho}\int_{-\infty}^{\infty} t' \,\mathbf{exp}\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t' + \frac{c^2}{3\rho^2}t'^3 + \frac{\boldsymbol{c}}{\rho}\xi\, t'^2\right]\right\} dt'$$

$$\mathbf{A}_{\perp}(\omega) = \psi\int_{-\infty}^{\infty} \mathbf{exp}\left\{ i\frac{\boldsymbol{\omega}}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\boldsymbol{\xi}^2\right)t' + \frac{c^2}{3\boldsymbol{\rho}^2}t'^3 + \frac{c}{\boldsymbol{\rho}}\boldsymbol{\xi}\, t'^2\right]\right\} dt'$$

To consider the effect of quadratic phase factor in these integrals, integrating the vector potential by part,

$$\mathbf{A}_{\mathrm{II}}(\omega) \cong \frac{c}{\rho}\int_{-\infty}^{\infty} t' \,\mathbf{exp}\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t' + \frac{c^2}{3\rho^2}t'^3 + \frac{c}{\rho}\xi\, t'^2\right]\right\} dt'$$

$$= \frac{c}{\rho}\int_{-\infty}^{\infty} t' \,\mathbf{exp}\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t' + \frac{c^2}{3\rho^2}t'^3\right]\right\}$$

$$\cdot\, \mathbf{exp}\left\{ i\frac{\omega}{2}\frac{c}{\rho}\xi\, t'^2\right\} dt'$$

Remember integral by part as,

$$\int f'(t')g(t')dt' = f(t')g(t') - \int f(t')g'(t')dt'$$

$$f'(t') = t'\exp\left\{i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t'+\frac{c^2}{3\rho^2}t'^3\right]\right\}$$

$$g(t') = \exp\left\{i\frac{\omega}{2}\frac{c}{\rho}\xi\, t'^2\right\}$$

Then,

$$f(t') = \int_{-\infty}^{\infty} t'\exp\left\{i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t'+\frac{c^2}{3\rho^2}t'^3\right]\right\}dt'$$

$$g'(t') = i\frac{\omega}{2}\xi\frac{c}{\rho}\frac{1}{2}t\cdot\exp\left\{i\frac{\omega}{2}\frac{c}{\rho}\xi\, t'^2\right\}$$

$$\mathbf{A}_{II}(\omega) \cong \frac{c}{\rho}\left[\int_{-\infty}^{\infty} t' \exp\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t' + \frac{c^2}{3\rho^2}\ t'^3\right]\right\} dt' \cdot \exp\left\{ i\frac{\omega}{2}\frac{c}{\rho}\xi\, t'^2\right\}\right.$$

$$\left. - \int_{-\infty}^{\infty} f(t') \cdot i\frac{\omega}{2}\xi\frac{c}{\rho}\frac{1}{2} t' \cdot \exp\left\{ i\frac{\omega}{2}\xi\frac{c}{\rho} t'^2\right\} dt'\right]$$

as mention in later, $f\,(t')$ is represented by a modified Bessel function $K_{2/3}(\omega,\psi,\xi)$, and will be not dependent to t', the second term will be,

$$\boldsymbol{i}\frac{\boldsymbol{\omega}}{2}\boldsymbol{\xi}\frac{\boldsymbol{c}}{\boldsymbol{\rho}}\frac{1}{2}\mathrm{a}\cdot \mathrm{K}_{2/3}\int_{-\infty}^{\infty} t'\cdot exp\left\{ i\frac{\boldsymbol{\omega}}{2}\boldsymbol{\xi}\frac{\boldsymbol{c}}{\boldsymbol{\rho}} t'^2\right\}\boldsymbol{d}t'$$

In here, a is constant.

In this term, the integral

$$\int_{-\infty}^{\infty} t' \cdot exp\left\{ i \frac{\boldsymbol{\omega}}{2} \boldsymbol{\xi} \frac{\boldsymbol{c}}{\boldsymbol{\rho}} t'^2 \right\} \boldsymbol{d}t'$$

is one dimensional Fresnel transform of odd function *t'*, and it will be zero. Then **A** will be,

$$\mathbf{A}_{\mathrm{II}}(\omega) \cong \frac{c}{\rho} \int_{-\infty}^{\infty} t' \, exp\left\{ i \frac{\omega}{2} \left[\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 \right) t' + \frac{\mathrm{c}^2}{3\rho^2} t'^3 \right] \right\} dt' \cdot exp\left\{ i \frac{\omega}{2} \frac{c}{\rho} \xi \, t'^2 \right\}$$

In this equation, the quadratic phase term

$$\mathbf{exp}\left\{ i\frac{\boldsymbol{\omega}}{2}\frac{\boldsymbol{c}}{\boldsymbol{\rho}}\boldsymbol{\xi}\, t'^2 \right\}$$

has unit amplitude. We will take absolute value of **A** in later, so we can replace this term with 1. Finally we can conclude **A** will be,

$$\mathbf{A}_{\mathrm{II}}(\omega) \cong \frac{c}{\rho}\int_{-\infty}^{\infty} t' \, exp\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 \right) t' + \frac{\mathrm{c}^2}{3\rho^2} t'^3 \right] \right\} dt'$$

For the vertical polarized component of **A**, using same way,

$$\boldsymbol{f'(t')} = \mathbf{exp}\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t' + \frac{c^2}{3\boldsymbol{\rho}^2}t'^3\right]\right\}$$

$$\boldsymbol{g(t')} = \mathbf{exp}\left\{ i\frac{\omega}{2}\frac{\boldsymbol{c}}{\rho}\xi\, t'^2\right\}$$

Then,

$$\boldsymbol{f}(\boldsymbol{t}') = \int_{-\infty}^{\infty} \mathbf{exp}\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 \right) t' + \frac{c^2}{3\rho^2} t'^3 \right] \right\} \boldsymbol{dt}'$$

$$\boldsymbol{g}'(\boldsymbol{t}') = \boldsymbol{i}\frac{\omega}{2}\xi\frac{\boldsymbol{c}}{\rho}\frac{1}{2}\boldsymbol{t} \cdot \mathbf{exp}\left\{ i\frac{\omega}{2}\frac{\boldsymbol{c}}{\rho}\xi\, t'^2 \right\}$$

$$\mathbf{A}_{\perp}(\omega) \cong \psi \int_{-\infty}^{\infty} \exp\left\{ i \frac{\omega}{2} \left[\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 \right) t' + \frac{c^2}{3\rho^2} t'^3 \right] \right\} dt' \cdot \exp\left\{ i \frac{\omega}{2} \frac{c}{\rho} \xi t'^2 \right\}$$

$$- \int_{-\infty}^{\infty} f(t') \cdot i \frac{\omega}{2} \xi \frac{c}{\rho} \frac{1}{2} t' \cdot \exp\left\{ i \frac{\omega}{2} \xi \frac{c}{\rho} t'^2 \right\} dt'$$

as mention in the case of $\xi = 0$, $f(t')$ is represented by a modified Bessel function $K_{1/3}(\omega, \psi, \xi)$, and will be not dependent to t', the second term will be,

$$i \frac{\omega}{2} \xi \frac{c}{\rho} \frac{1}{2} \mathrm{a} \cdot K_{1/3} \int_{-\infty}^{\infty} t' \cdot exp\left\{ i \frac{\omega}{2} \xi \frac{c}{\rho} t'^2 \right\} dt'$$

In here, a is constant.

Same way as in horizontal, In this term, the integral

$$\int_{-\infty}^{\infty} t' \cdot exp\left\{ i\frac{\omega}{2}\xi\frac{c}{\rho}t'^2 \right\} dt'$$

is one dimensional Fresnel transform of odd function *t'*, and it will be zero. Then **A** will be,

$$\mathbf{A}_{II}(\omega) \cong \psi \int_{-\infty}^{\infty} exp\left\{ \mathrm{i}\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\mathrm{t}'+\frac{\mathrm{c}^2}{3\rho^2}\mathrm{t}'^3\right]\right\} dt' \cdot exp\left\{ \mathrm{i}\frac{\omega}{2}\frac{c}{\rho}\xi\,\mathrm{t}'^2 \right\}$$

In the same manner in horizontal, the quadratic phase term

$$\mathbf{exp}\left\{ i\frac{\boldsymbol{\omega}}{2}\frac{\boldsymbol{c}}{\boldsymbol{\rho}}\boldsymbol{\xi}\, t'^2 \right\}$$

has unit amplitude. We will take absolute value of **A** in later, so we can replace this term with 1. Finally we can conclude **A** will be,

$$\mathbf{A}_{\perp}(\omega) \cong \psi \int_{-\infty}^{\infty} \exp\left\{ i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)t' + \frac{c^2}{3\rho^2}t'^3\right]\right\} dt'$$

The spectrum intensity distribution I is given by,

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \mathbf{A}_{\mathrm{II}}(\omega) + \mathbf{A}_{\perp}(\omega) \right|^2$$

A change of variable to

$$x = \frac{ct'}{\rho\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}}$$

them,

$$t' = \frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}} x$$

$$dt' = \frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}} dx$$

$$\zeta = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{3}{2}}$$

the integrals in $\mathbf{A}_{\perp}(\omega)$ into;

$$\mathbf{A}_{\mathrm{II}}(\omega)=\frac{c}{\rho}\cdot\left(\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}\right)\int_{-\infty}^{\infty}x\,exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}\frac{\rho}{c}x\right)\right.$$

$$\left.+\frac{\mathrm{c}^2}{3\rho^2}\frac{\rho^3}{c^3}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}\right]\cdot\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}dx$$

$$=\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\int_{-\infty}^{\infty}x\,exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{3}{2}}\frac{\rho}{c}x\right)+\frac{\rho}{3c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{3}{2}}\right]dx$$

the integrals in $\mathbf{A}_{\mathrm{II}}(\omega)$ into;

$$\mathbf{A}_{\mathrm{II}}(\omega)=\psi\cdot\int_{-\infty}^{\infty}x\,exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}\frac{\rho}{c}x\right)\right.$$

$$\left.+\frac{\mathrm{c}^2}{3\rho^2}\frac{\rho^3}{c^3}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\cdot\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}\right]\cdot\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}dx$$

$$=\psi\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}x\,exp\left[i\frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{3}{2}}\frac{\rho}{c}x\right)+\frac{\rho}{3c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{3}{2}}\right]dx$$

Introducing following parameter ζ ;

$$\zeta = \frac{\omega\rho}{3\mathrm{c}}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^{\frac{3}{2}}$$

Then, we can transform the integrals in $\mathbf{A}_{\parallel}(\omega)$ and $\mathbf{A}_{\perp}(\omega)$ into the form;

$$\mathbf{A}_{\parallel}(\omega)=\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\int_{-\infty}^{\infty} x\ exp\left\{\mathrm{i}\frac{3}{2}\zeta\left[\ x+\frac{1}{3}x^3\right]\right\}\mathrm{d}x$$

$$\mathbf{A}_{\perp}(\omega)=\psi\,\frac{\rho}{c}\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)\int_{-\infty}^{\infty} exp\left\{\mathrm{i}\frac{3}{2}\zeta\left[\ x+\frac{1}{3}x^3\right]\right\}\mathrm{d}x$$

These integrals are identifiable as Airy integrals, and alternatively as modified Bessel functions;

$$\int_0^\infty x\, sin\left(\frac{3}{2}\zeta\left(x+\frac{1}{3}x^3\right)\right)dx = \frac{1}{\sqrt{3}}K_{\frac{2}{3}}\left(\zeta\right)$$

$$\int_0^\infty cos\left(\frac{3}{2}\zeta\left(x+\frac{1}{3}x^3\right)\right)dx = \frac{1}{\sqrt{3}}K_{\frac{1}{3}}\left(\zeta\right)$$

Consequently the intensity radiated per unit frequency interval per unit solid angle is given by;

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^2\left[K_{2/3}^2(\zeta)+\frac{\psi^2}{\frac{1}{\gamma^2}+\psi^2+\xi^2}K_{1/3}^2(\zeta)\right]$$

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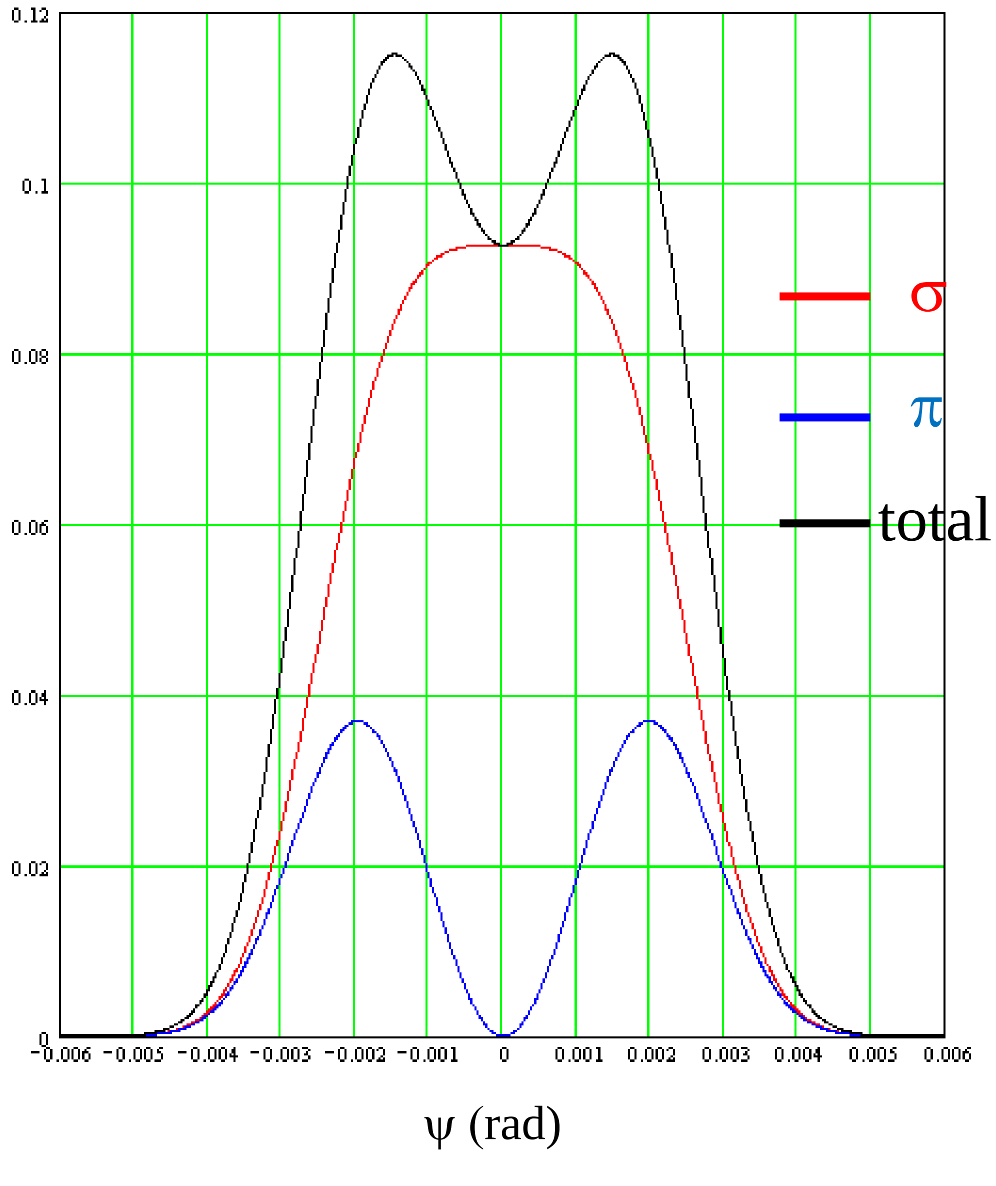
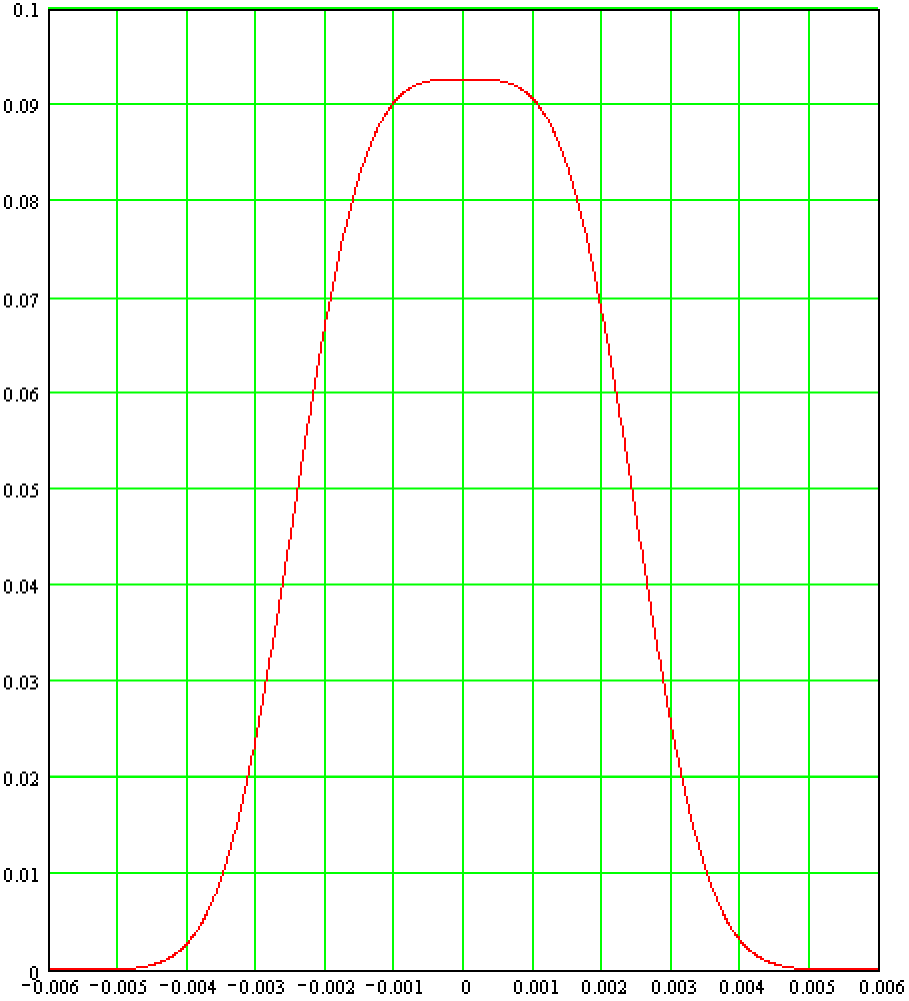


For horizontal direction, let put $\psi = 0$;

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\xi^2\right)^2\left[K_{2/3}^2(\zeta)\right]$$

Australian Synchrotron 500nm



ζ (rad)

This phenomenon is come from temporal squeezing factor in Exponent,

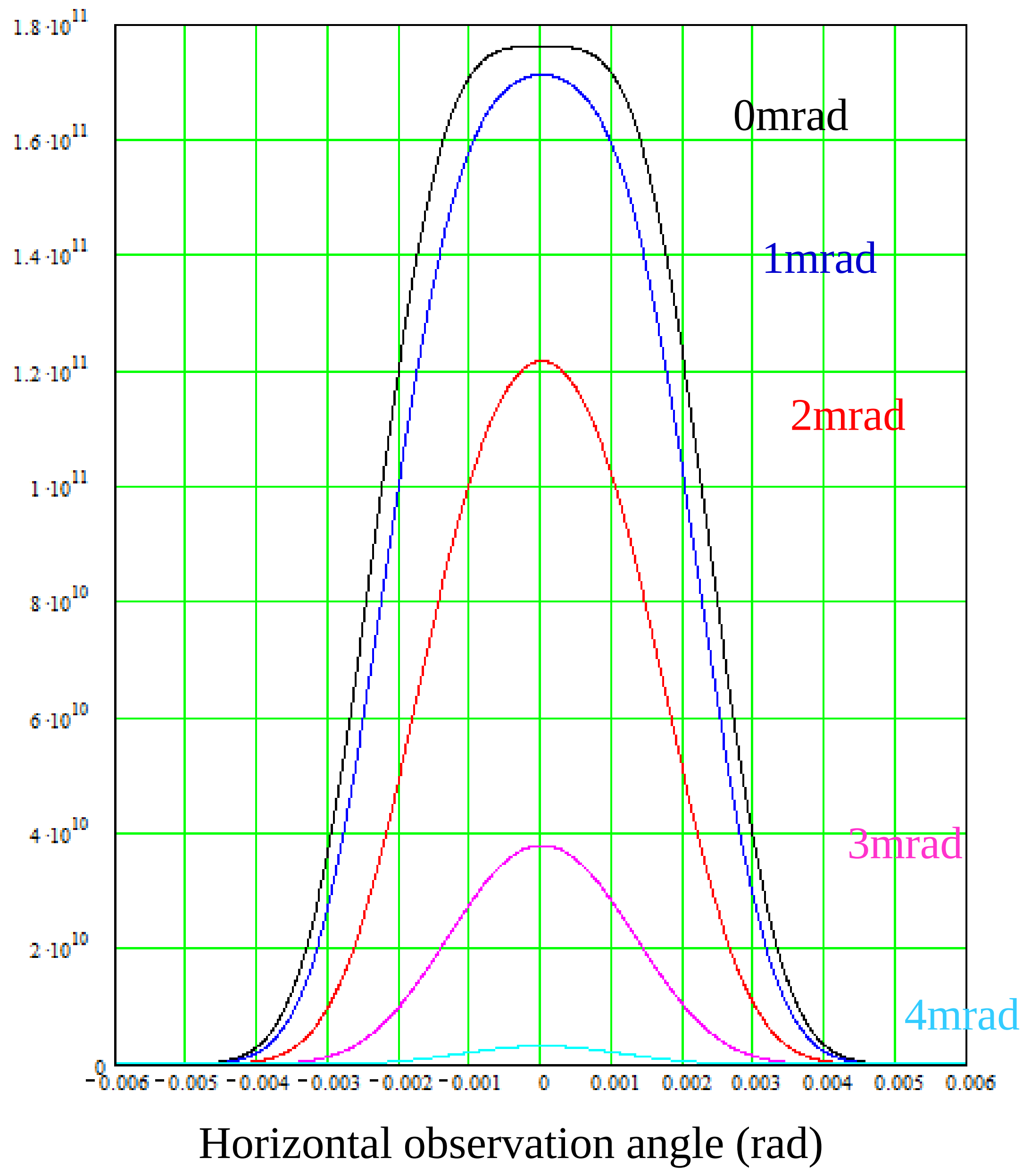
$$k(t') = \frac{dt(t')}{dt'} = \frac{1}{2}\left(\frac{1}{\gamma^2} + \psi^2 + \xi^2 + (\omega_0 t')^2 \right)$$

This has not only vertical observation angle ψ, but also horizontal observation angle ξ.

horizontal instantaneous opening at several vertical angle

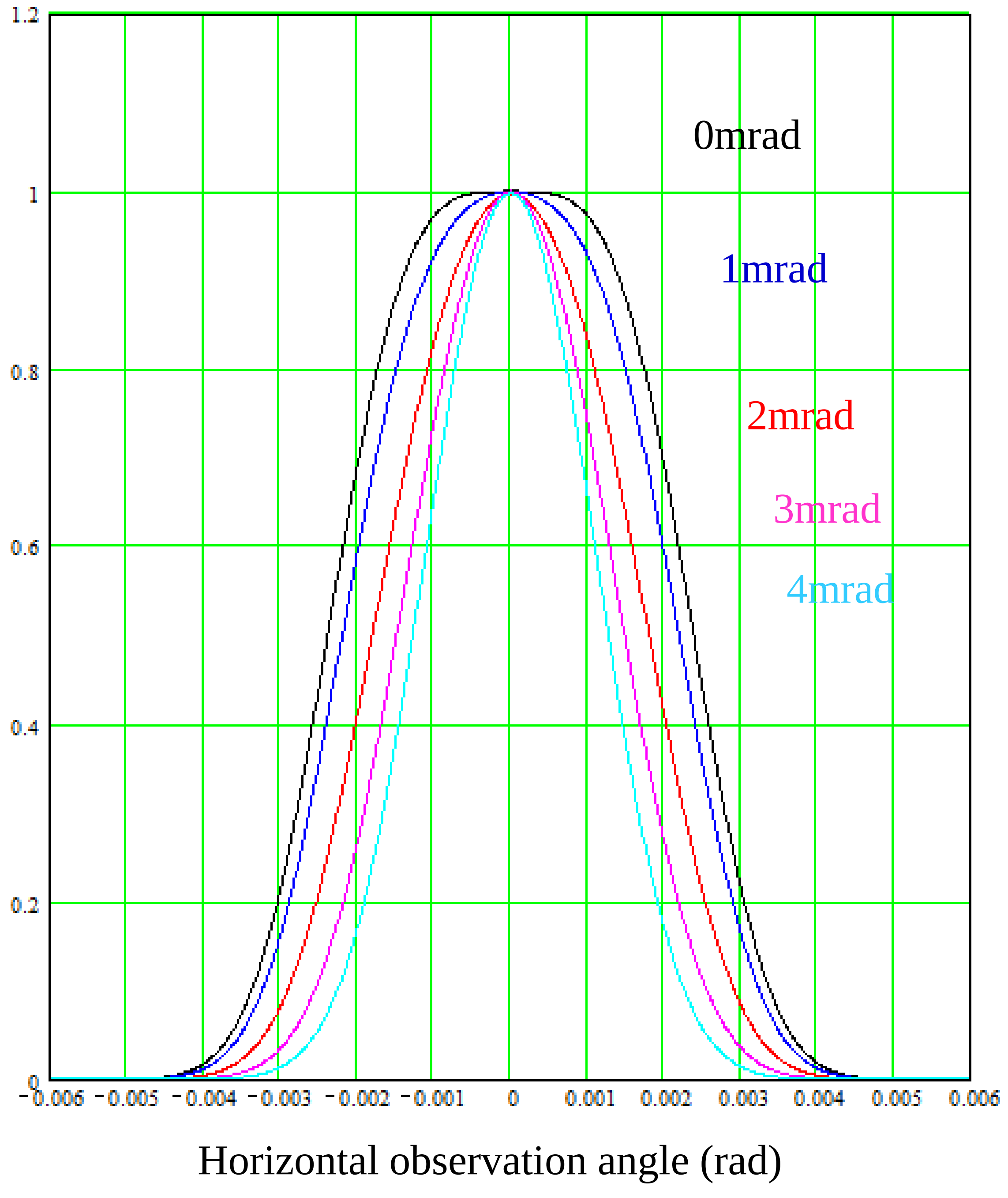
This calculation is based on SPEAR 3

condition



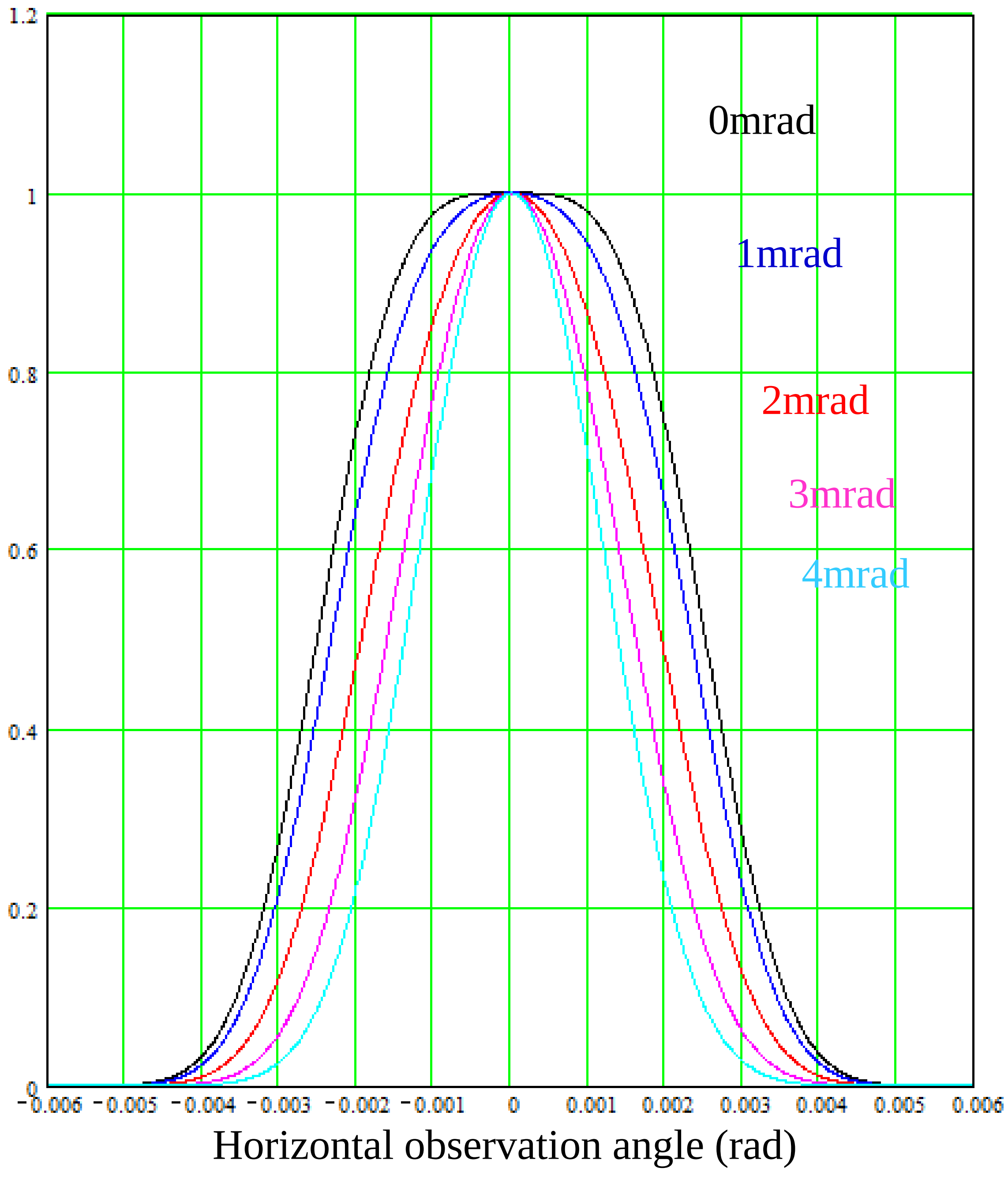
Normalized horizontal instantaneous opening at several vertical angle

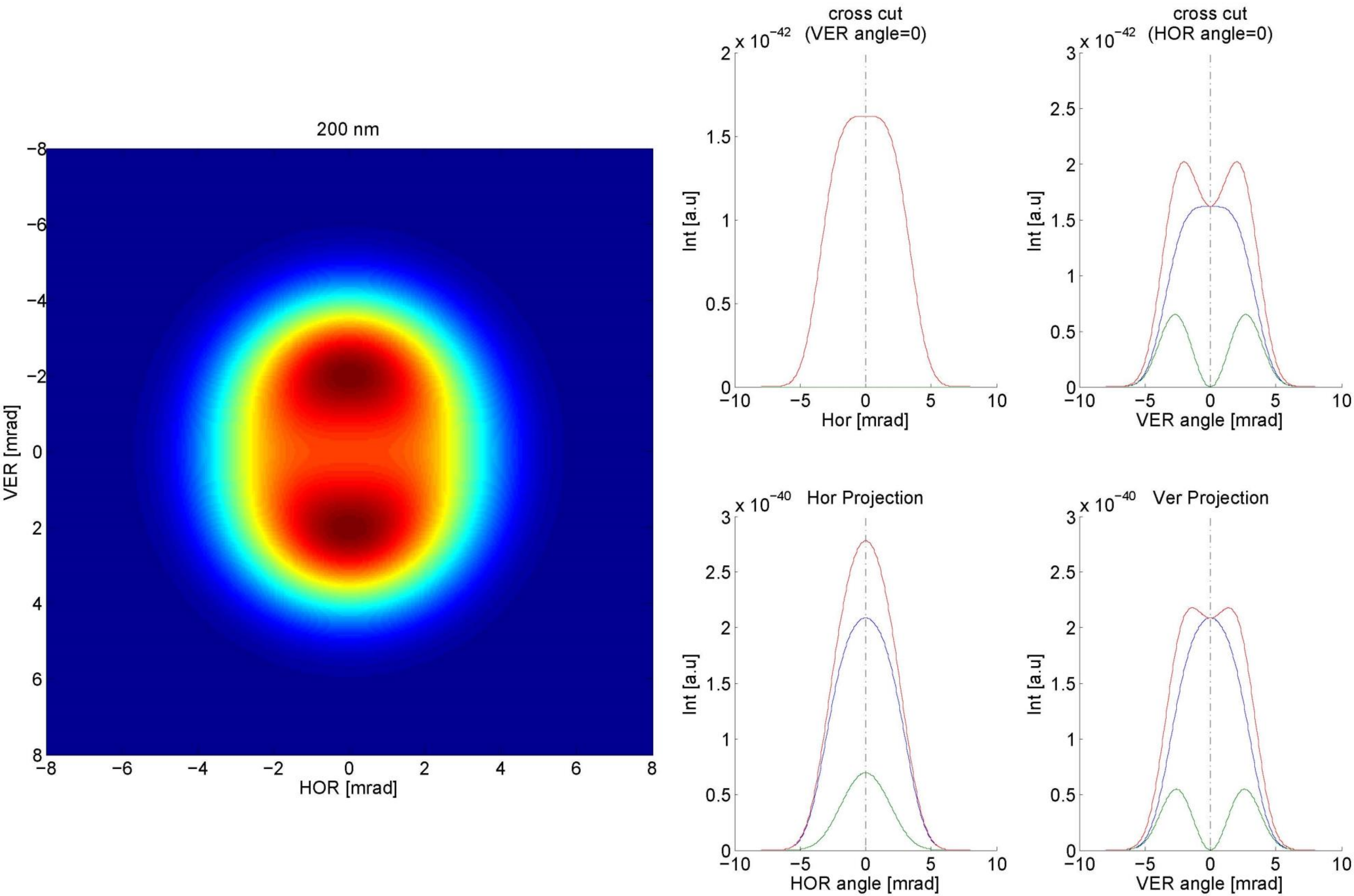
For SPEAR 3



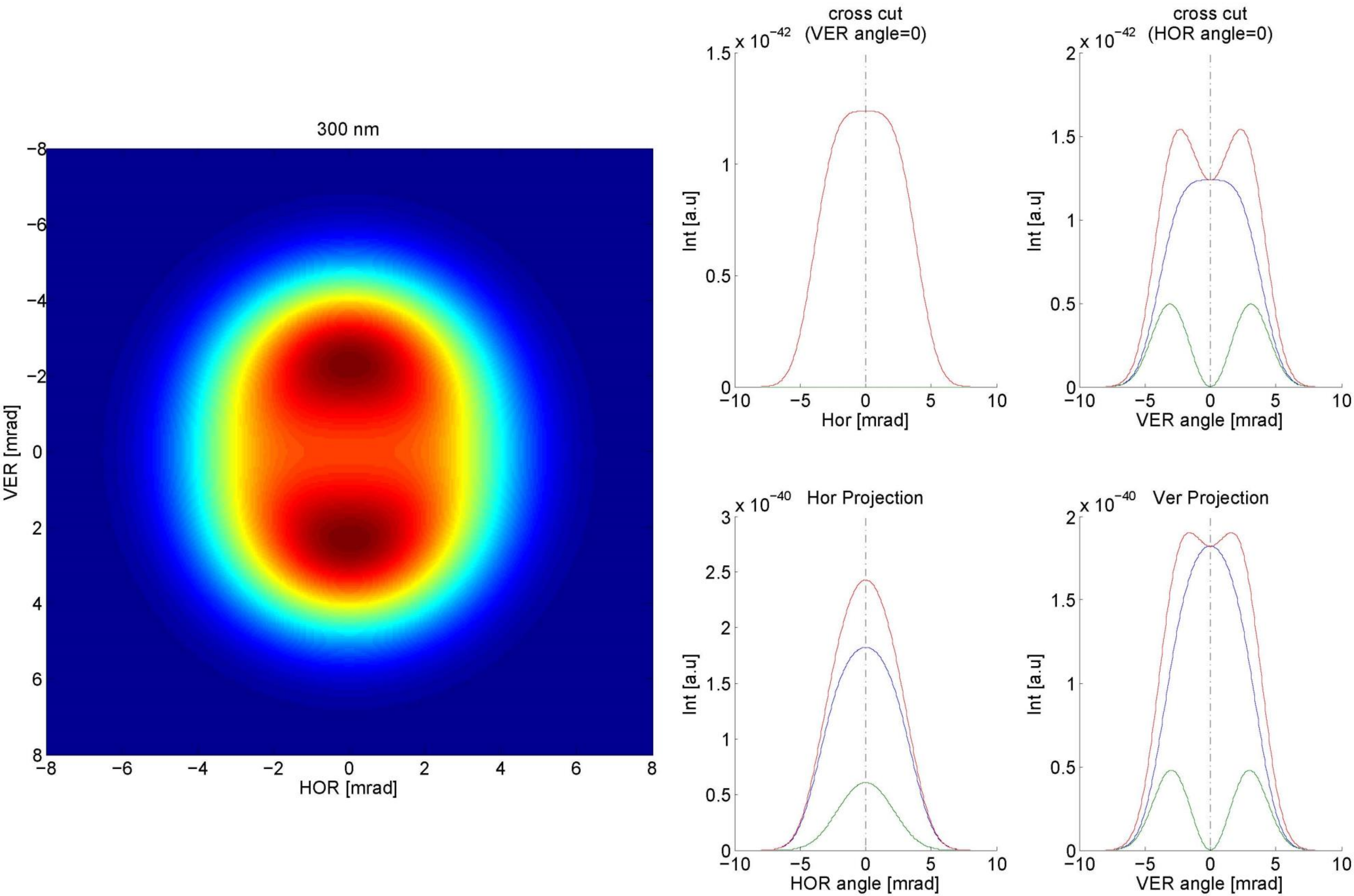
Normalized horizontal instantaneous opening at several vertical angle

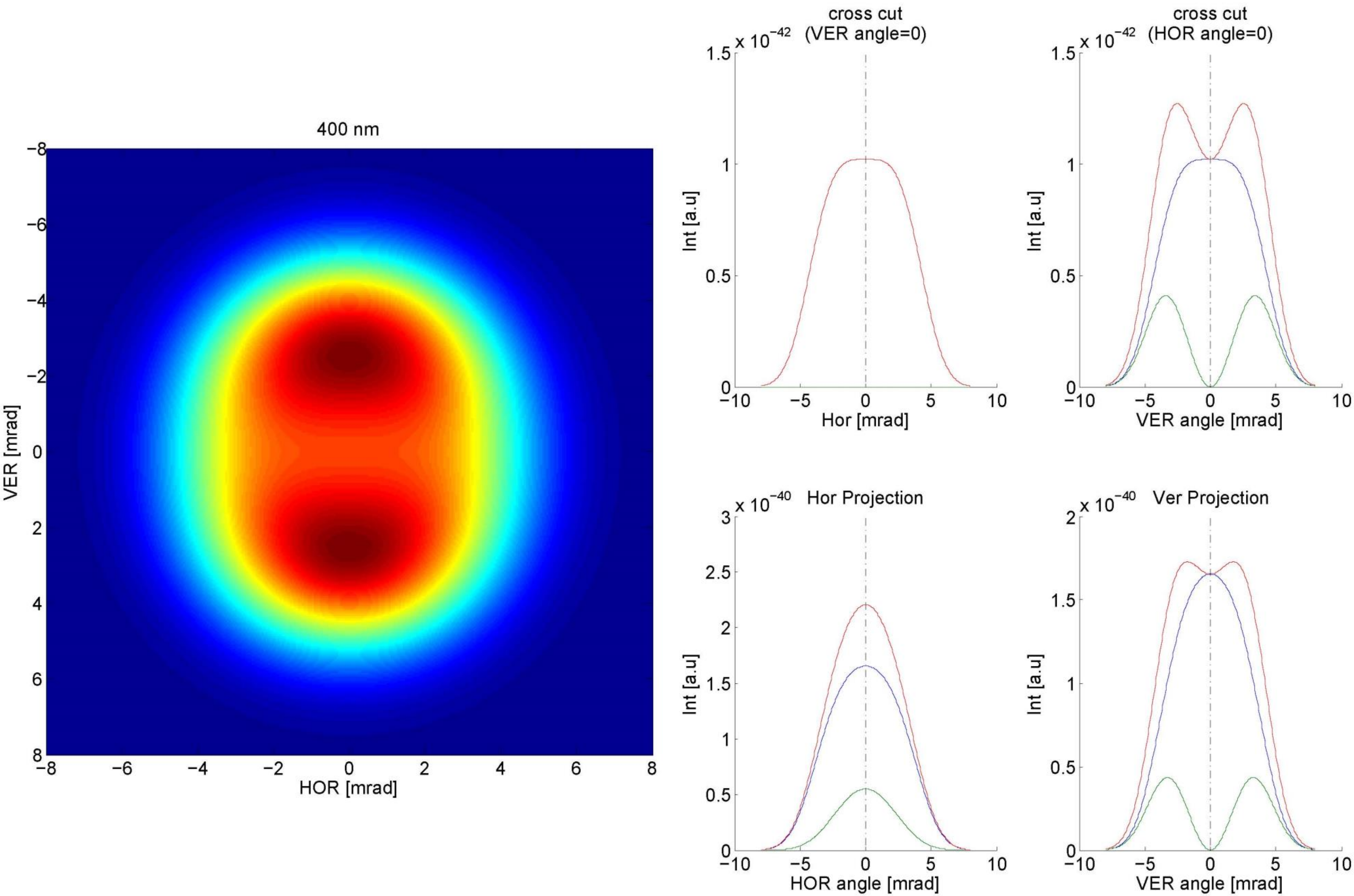
For ASLS



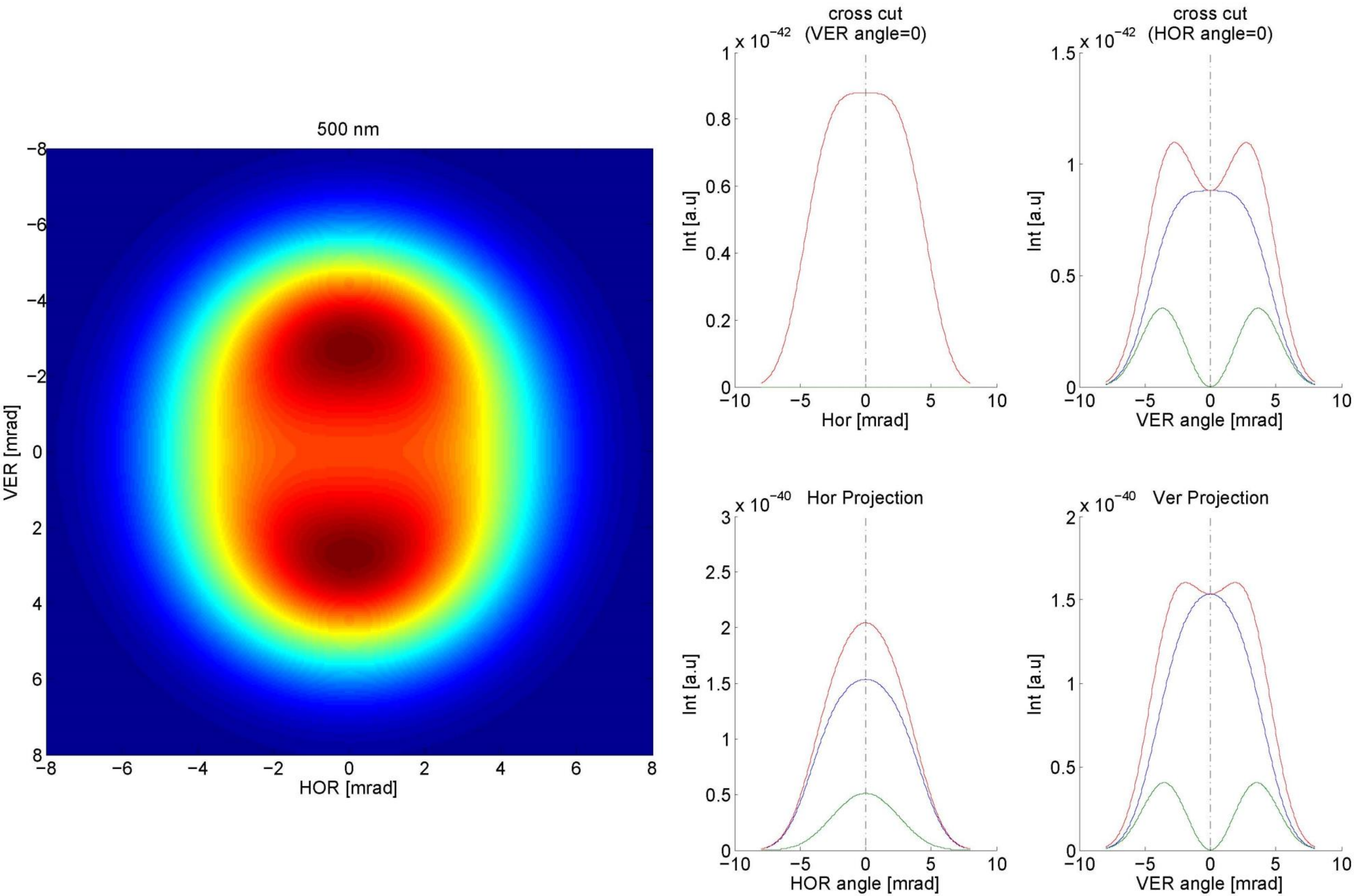


200 nm
VER [mrad]
HOR [mrad]
cross cut
x 10^-42 (VER angle=0)
Int [a.u]
Hor [mrad]
cross cut
x 10^-42 (HOR angle=0)
Int [a.u]
VER angle [mrad]
x 10^-40 Hor Projection
Int [a.u]
HOR angle [mrad]
x 10^-40 Ver Projection
Int [a.u]
VER angle [mrad]

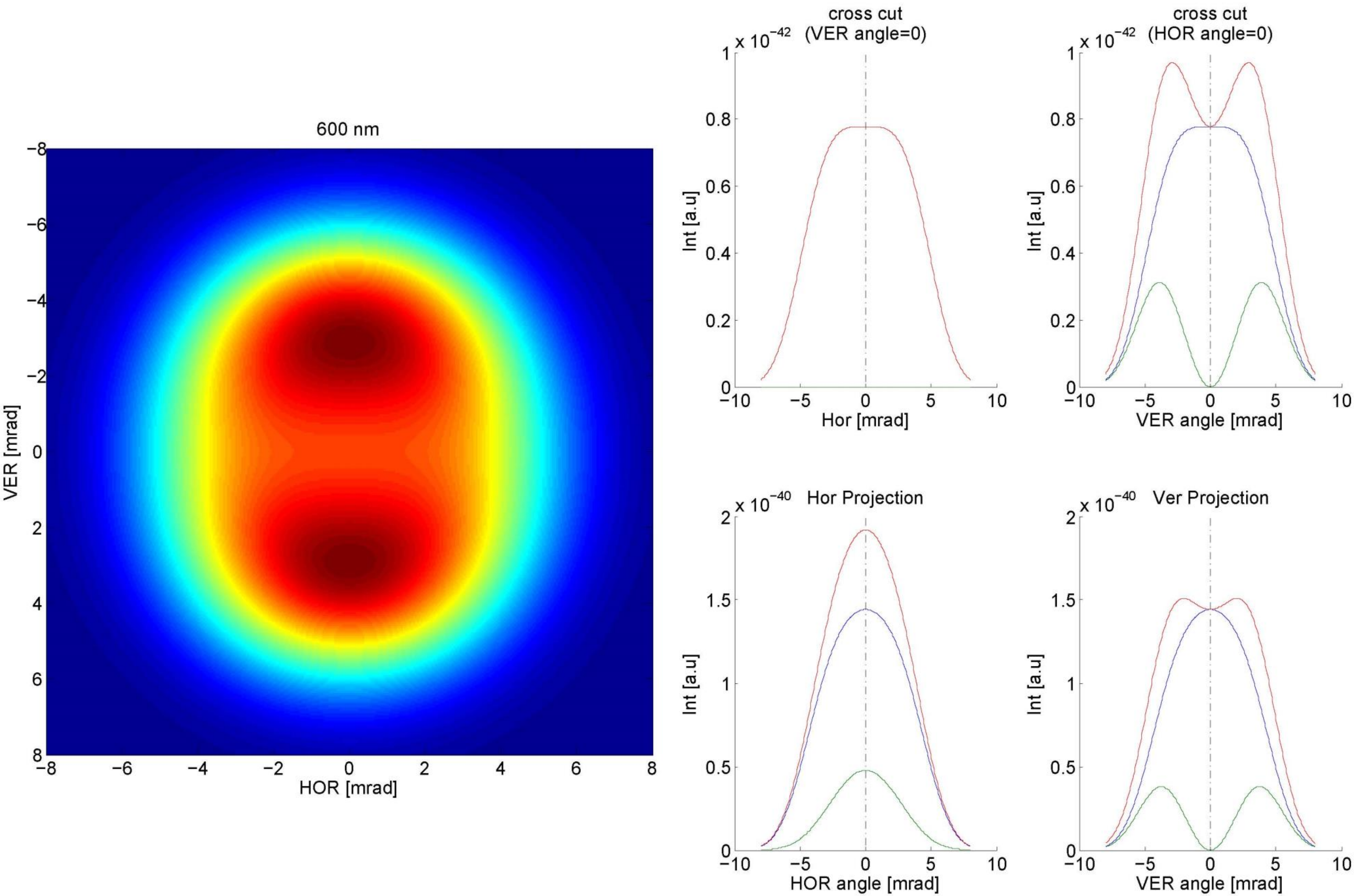

300 nm
VER [mrad]
HOR [mrad]
cross cut (VER angle=0)
x 10^-42
Int [a.u]
Hor [mrad]
cross cut (HOR angle=0)
x 10^-42
Int [a.u]
VER angle [mrad]
Hor Projection
x 10^-40
Int [a.u]
HOR angle [mrad]
Ver Projection
x 10^-40
Int [a.u]
VER angle [mrad]



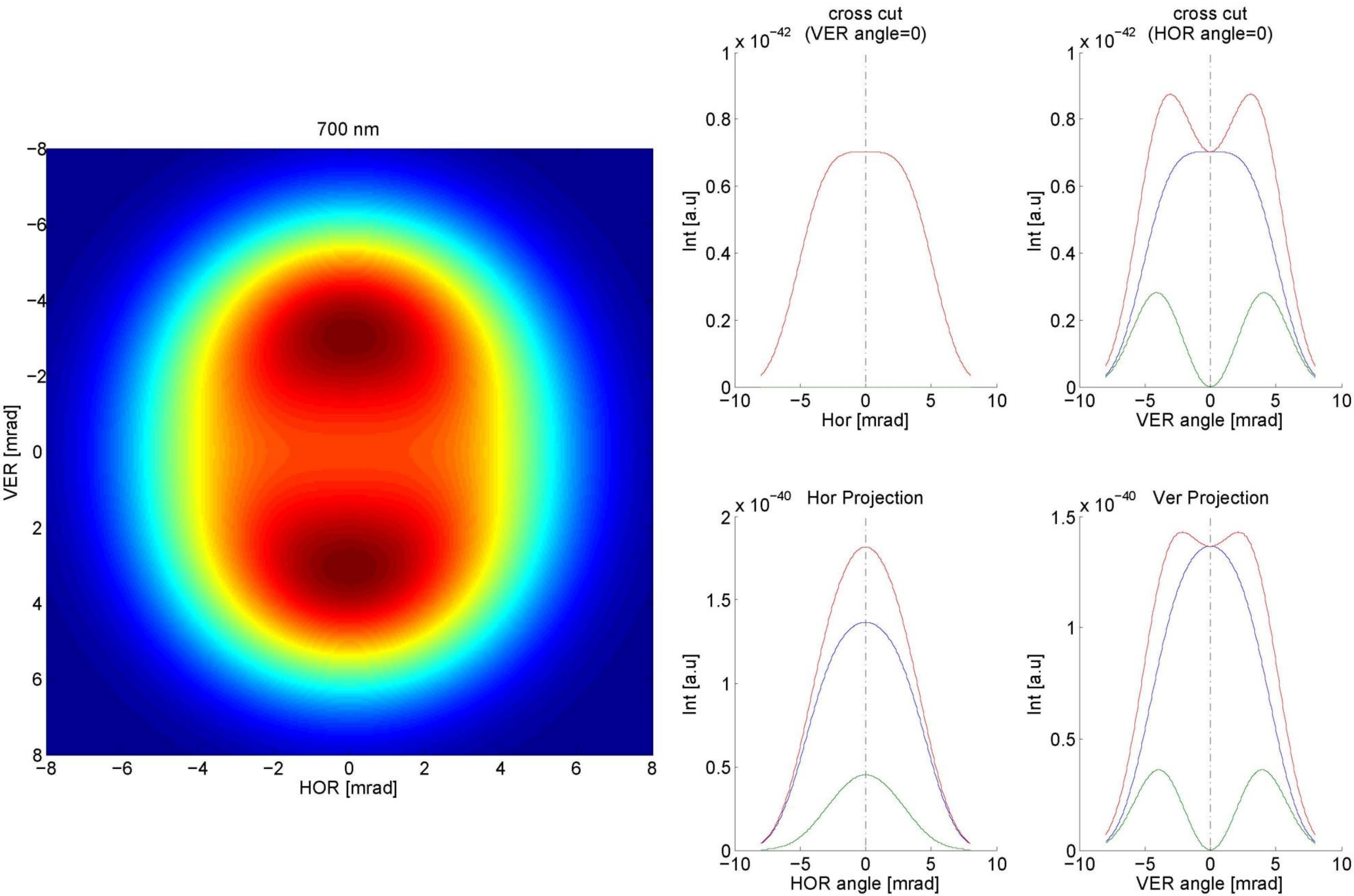
400 nm
VER [mrad]
HOR [mrad]
cross cut (VER angle=0)
x 10^-42
Int [a.u]
Hor [mrad]
cross cut (HOR angle=0)
x 10^-42
Int [a.u]
VER angle [mrad]
x 10^-40 Hor Projection
Int [a.u]
HOR angle [mrad]
x 10^-40 Ver Projection
Int [a.u]
VER angle [mrad]

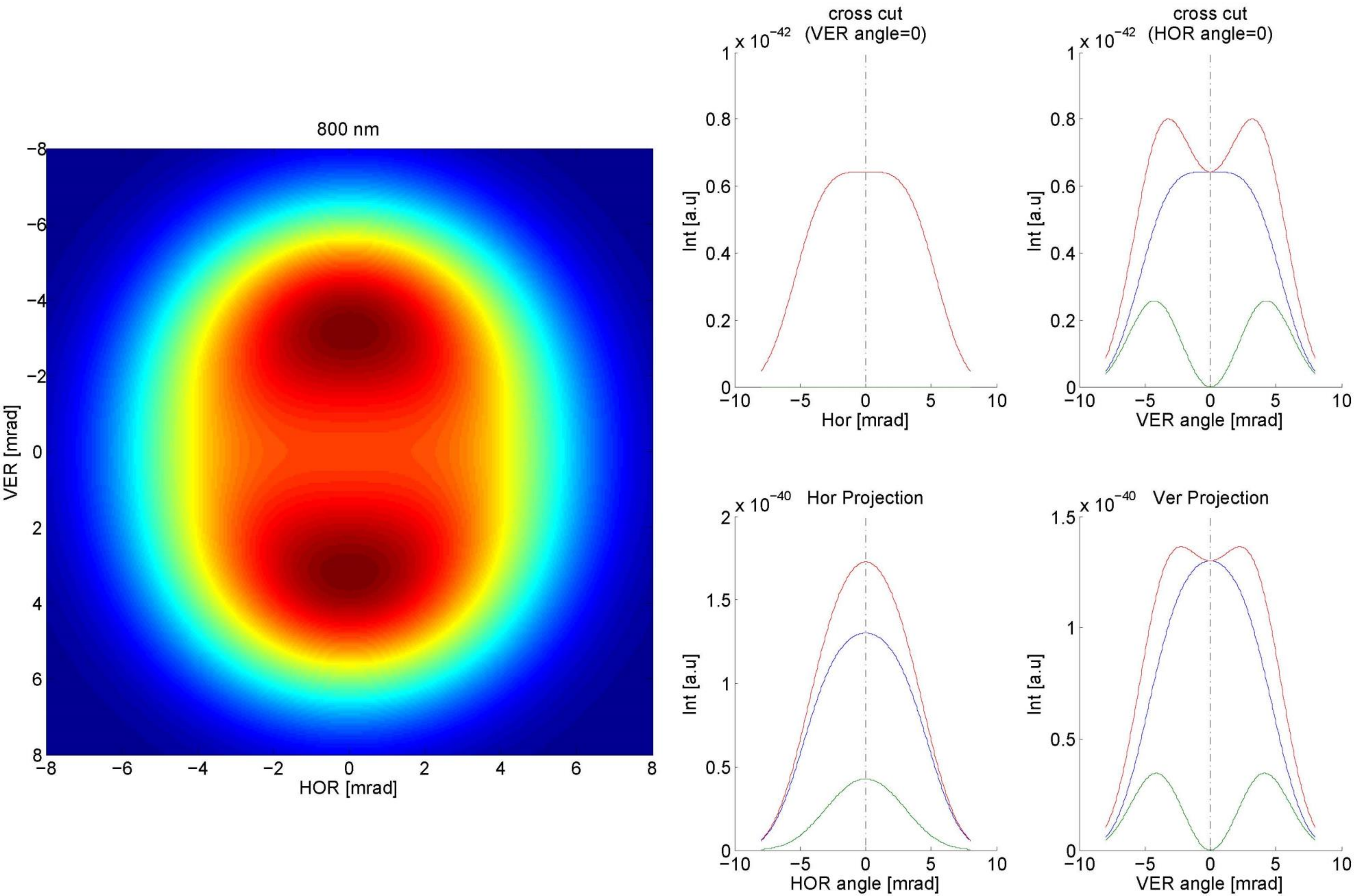


500 nm
VER [mrad]
HOR [mrad]
cross cut
(VER angle=0)
x 10^-42
Int [a.u]
Hor [mrad]
cross cut
(HOR angle=0)
x 10^-42
Int [a.u]
VER angle [mrad]
Hor Projection
x 10^-40
Int [a.u]
HOR angle [mrad]
Ver Projection
x 10^-40
Int [a.u]
VER angle [mrad]



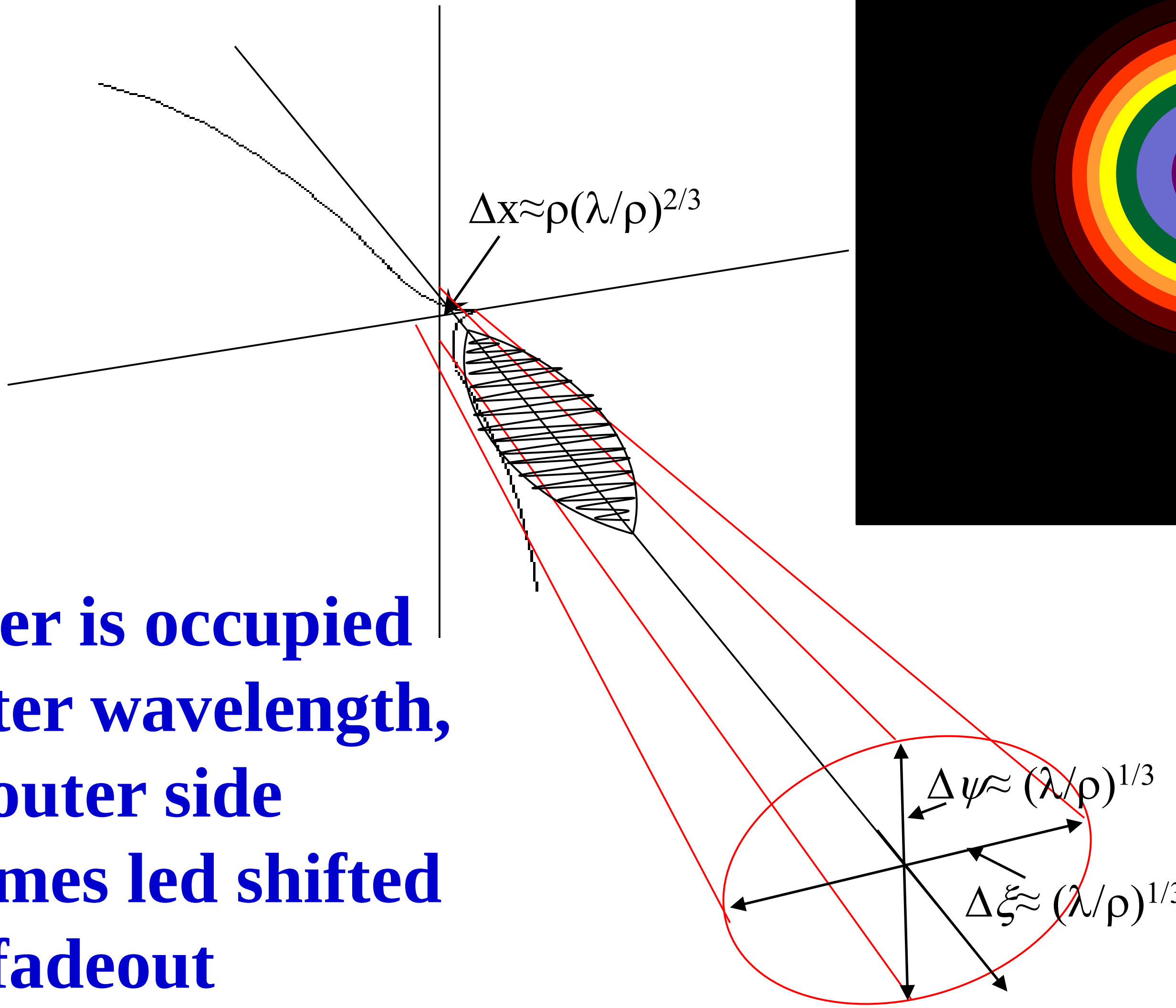
600 nm
VER [mrad]
HOR [mrad]
cross cut
x 10^-42 (VER angle=0)
Int [a.u]
Hor [mrad]
cross cut
x 10^-42 (HOR angle=0)
Int [a.u]
VER angle [mrad]
x 10^-40 Hor Projection
Int [a.u]
HOR angle [mrad]
x 10^-40 Ver Projection
Int [a.u]
VER angle [mrad]

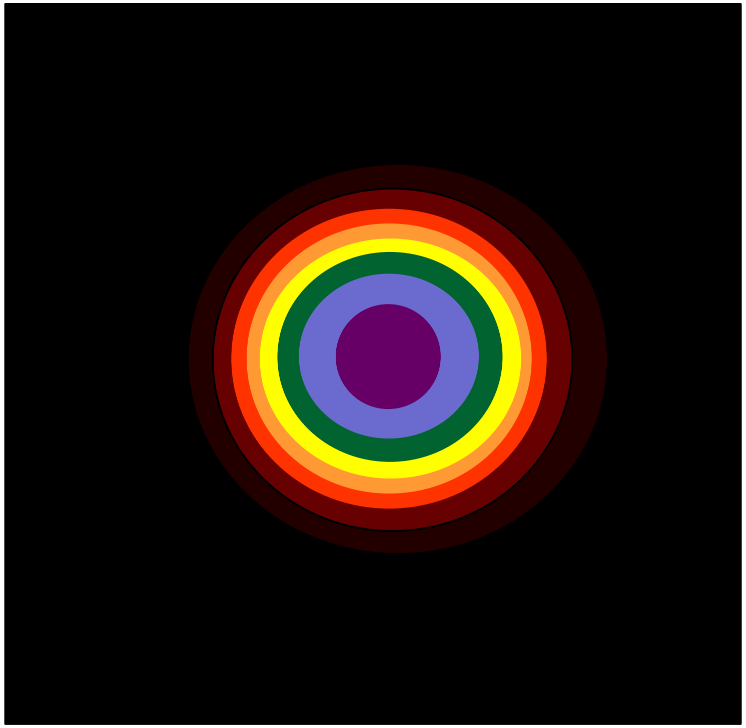

700 nm
VER [mrad]
HOR [mrad]
cross cut (VER angle=0)
x 10^-42
Int [a.u]
Hor [mrad]
cross cut (HOR angle=0)
x 10^-42
Int [a.u]
VER angle [mrad]
Hor Projection
x 10^-40
Int [a.u]
HOR angle [mrad]
Ver Projection
x 10^-40
Int [a.u]
VER angle [mrad]



800 nm
VER [mrad]
HOR [mrad]
cross cut (VER angle=0)
x 10^-42
Int [a.u]
Hor [mrad]
cross cut (HOR angle=0)
x 10^-42
Int [a.u]
VER angle [mrad]
Hor Projection
x 10^-40
Int [a.u]
HOR angle [mrad]
Ver Projection
x 10^-40
Int [a.u]
VER angle [mrad]

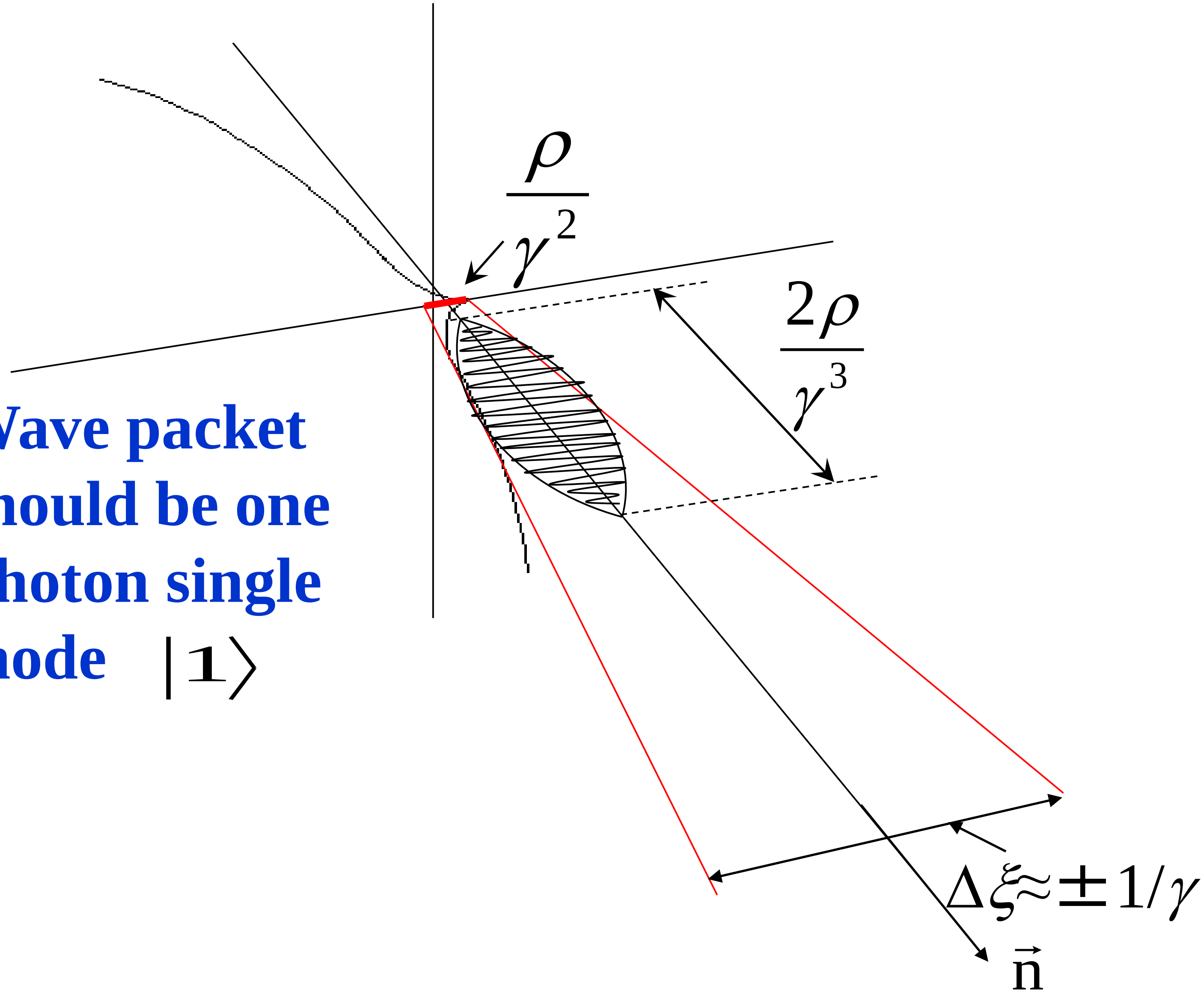
Qualitative understanding according to K.J. Kim's paradigm



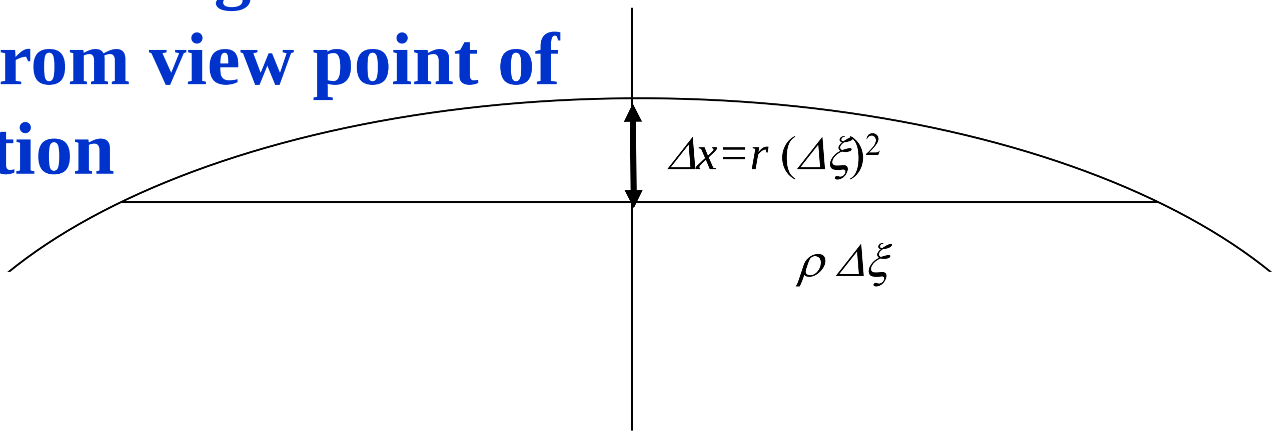


Center is occupied shorter wavelength, and outer side becomes led shifted and fadeout

Wave packet should be one photon single mode $|1\rangle$



Understanding of this result from view point of diffraction



According to diffraction, for the horizontal plane,

$$\Delta\xi \approx \frac{\lambda}{\Delta x} \approx \left(\frac{\lambda}{\rho}\right)^{1/3}$$

$$\Delta x \approx \rho \left(\frac{\lambda}{\rho}\right)^{2/3}$$

When $\lambda=\lambda_c$,

$$\lambda_c = \frac{\rho}{\gamma^3}$$

$$\Delta x_c \approx \frac{\lambda_c}{\Delta\xi} = \gamma\,\lambda_c = \frac{\rho}{\gamma^2}$$

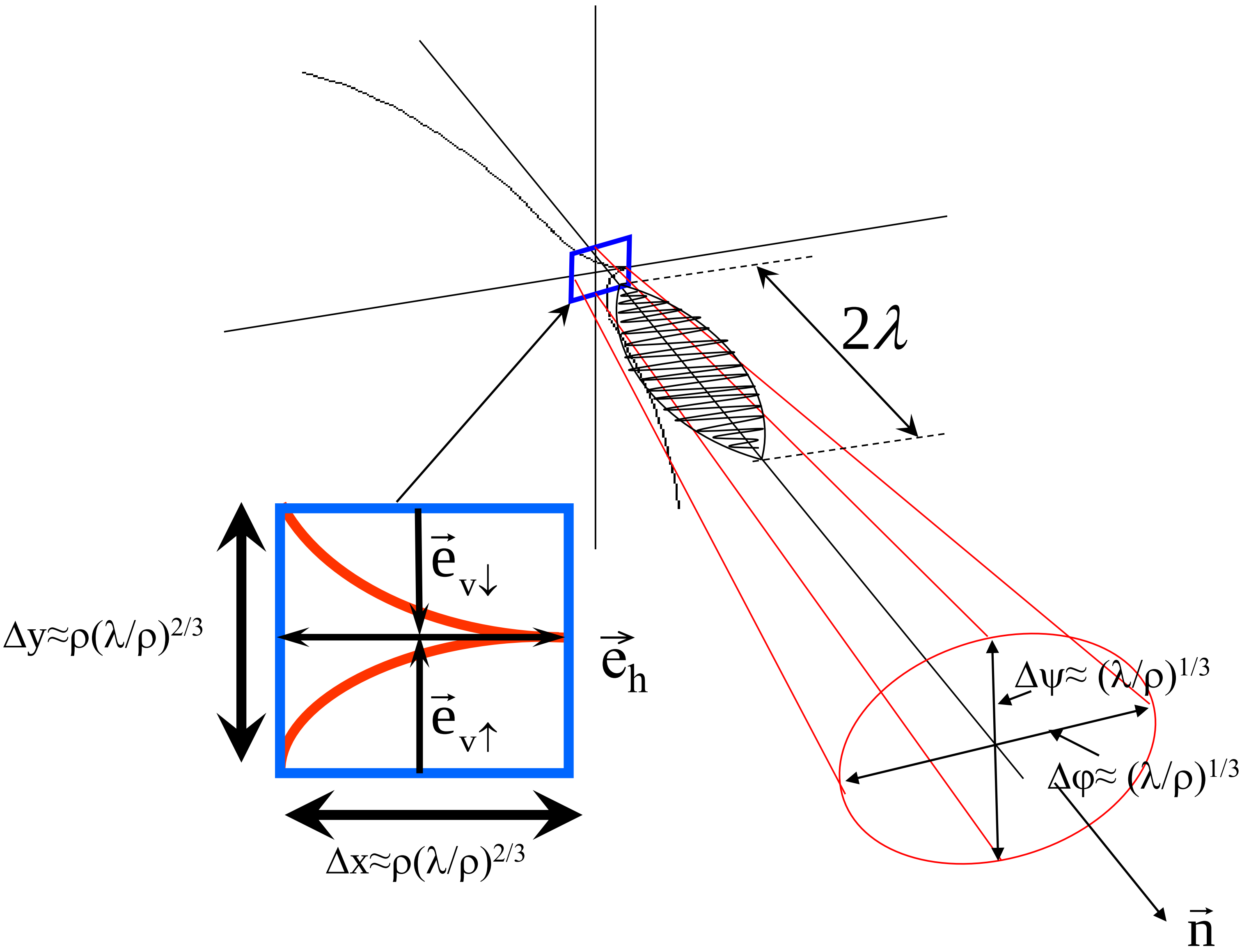
But how to understand in vertical plane?

Same way in horizontal, we should have same source size in vertical.

$$\Delta\psi \approx \frac{\lambda}{\Delta y} \approx \left(\frac{\lambda}{\rho}\right)^{1/3}$$

$$\Delta y \approx \rho\Delta\varphi\Delta\psi$$

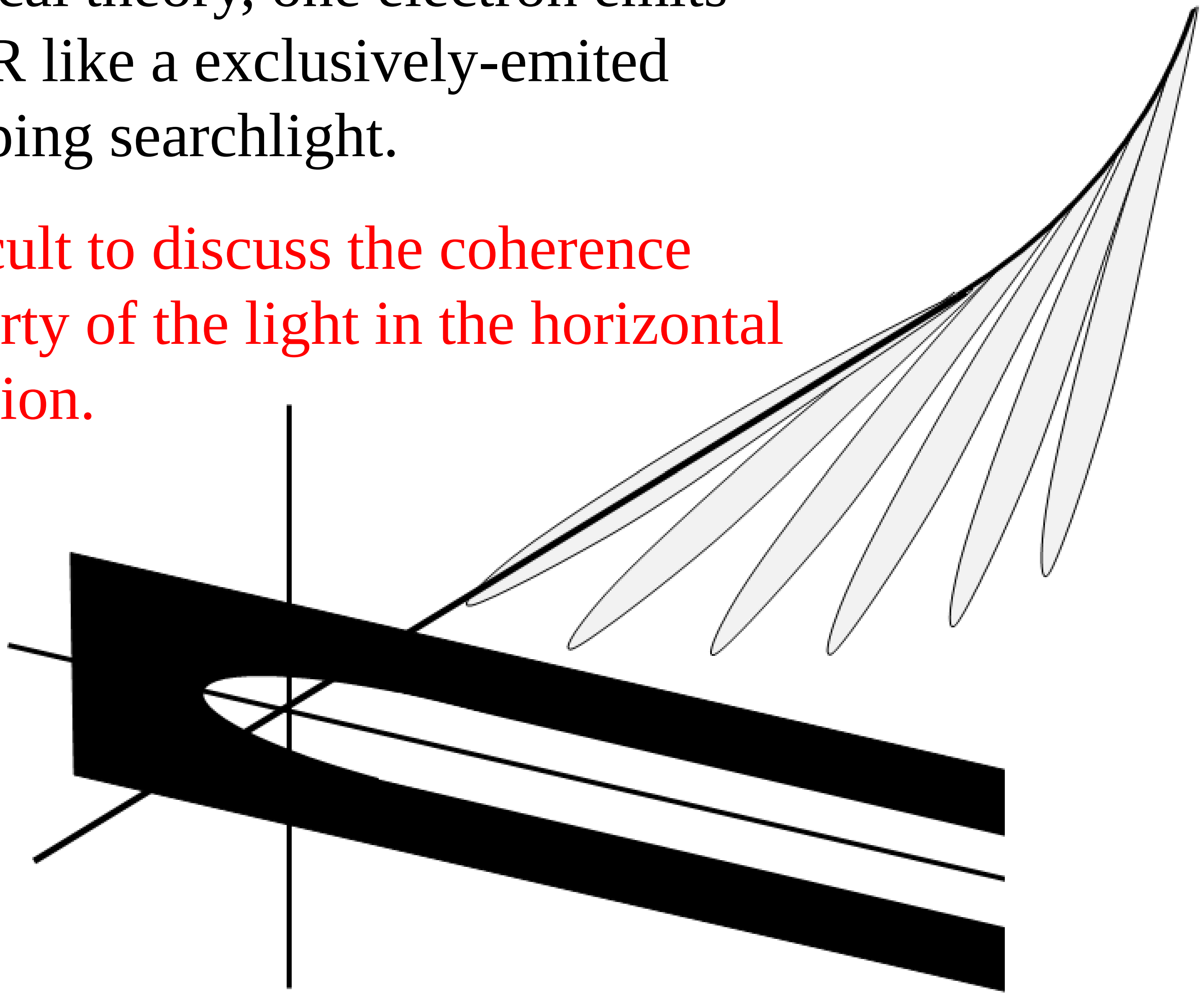
$$\Delta\, y \approx \rho\left(\frac{\lambda}{\rho}\right)^{2/3}$$



2λ
$\vec{e}_{v\downarrow}$
$\vec{e}_h$
$\vec{e}_{v\uparrow}$
Δy≈ρ(λ/ρ)^{2/3}
Δx≈ρ(λ/ρ)^{2/3}
Δψ≈ (λ/ρ)^{1/3}
Δφ≈ (λ/ρ)^{1/3}
$\vec{n}$

As a result from the point of view on classical theory, one electron emits the SR like a exclusively-emited sweeping searchlight.

Difficult to discuss the coherence property of the light in the horizontal direction.



Treatment of synchrotron radiation is completely classical, but the language of "Photon" can be used. The number of photons per unit frequency and unit solid angle is obtained by dividing intensity distribution by $\hbar\omega$. And using fine-structure constant α (it is combination constant of electromagnetic interaction)

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^2\left[K_{2/3}^2(\zeta)+\frac{\psi^2}{\frac{1}{\gamma^2}+\psi^2+\xi^2}K_{1/3}^2(\zeta)\right]$$

Introducing form factor *f*,

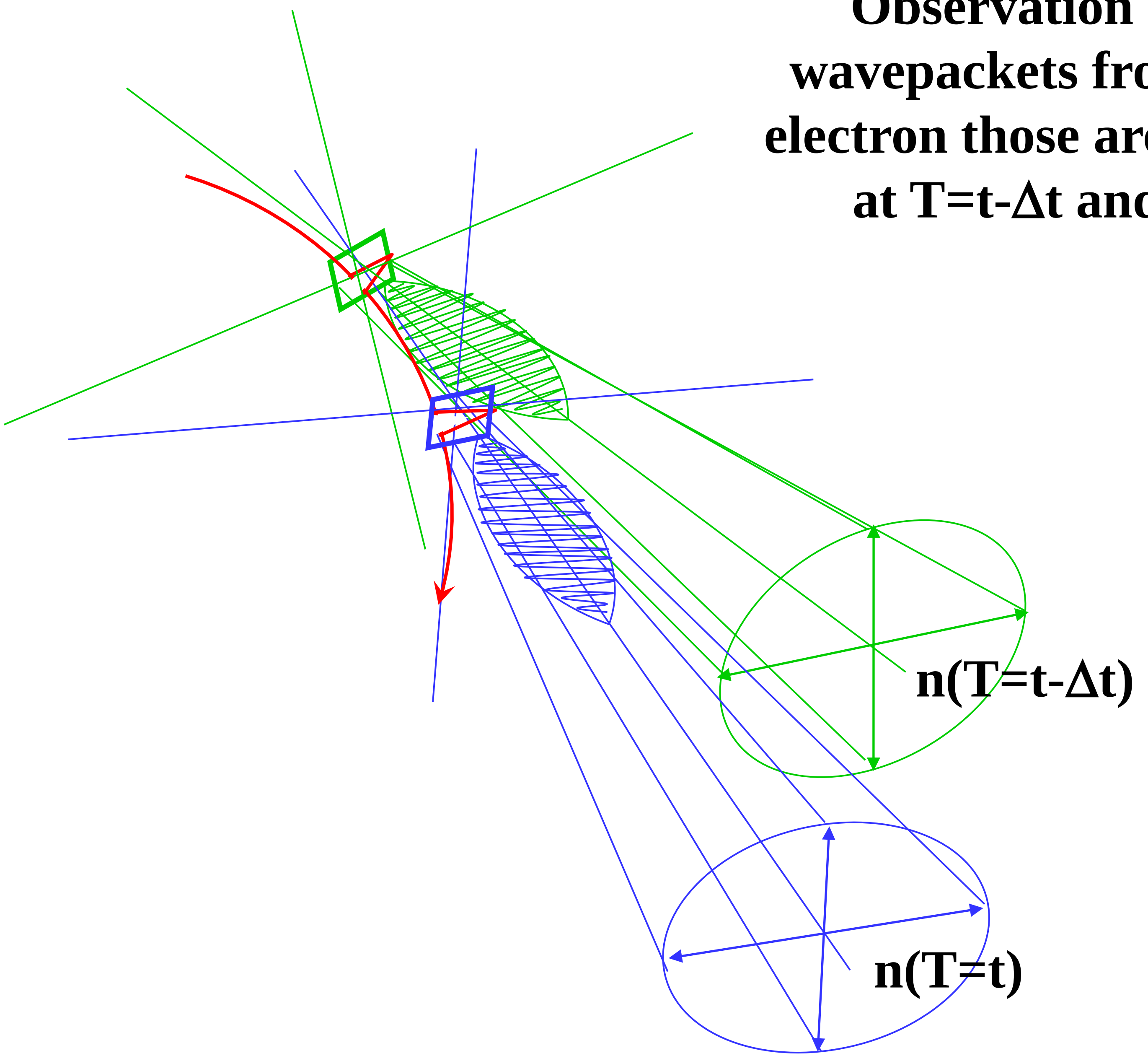
$$f\left(\omega,\gamma,\rho,\psi,\xi\right)=\frac{\omega}{3\pi^2}\left(\frac{\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\psi^2+\xi^2\right)^2\left[K_{2/3}^2\left(\zeta\right)+\frac{\psi^2}{\frac{1}{\gamma^2}+\psi^2+\xi^2}K_{1/3}^2\left(\zeta\right)\right]$$

The number of photons per unit frequency and unit solid angle is

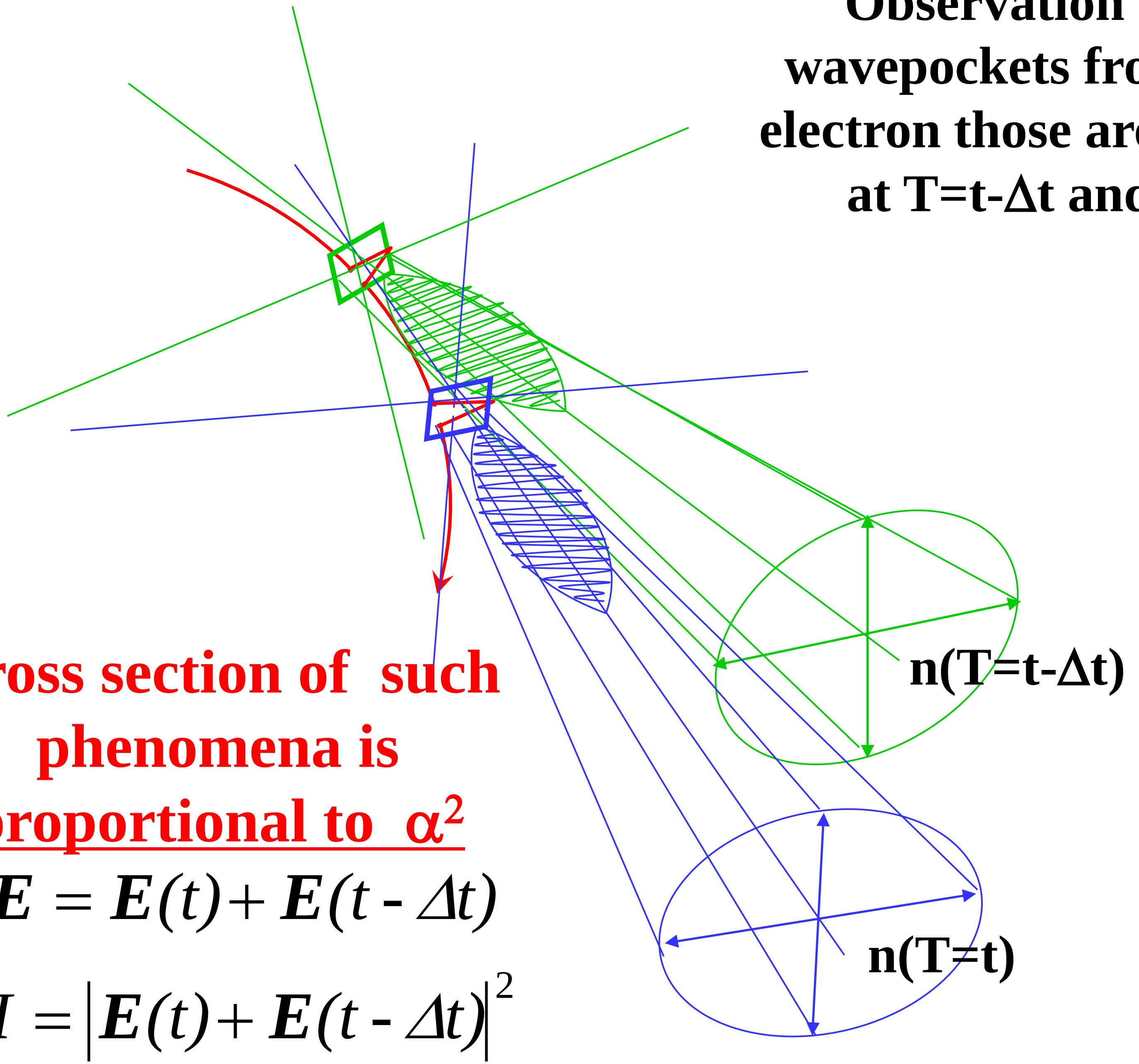
$$\frac{d^2N}{d\omega d\Omega}=\alpha\ f\left(\omega,\gamma,\rho,\psi,\xi\right)$$

So, roughly speaking, with the probability of 1/137=0.0072, photon are emitted to unit frequency and unit solid angle from single particle.

Observation of two wavepackets from single electron those are radiated at T=t-Δt and T=t .



Observation of two wavepockets from single electron those are radiated at T=t-Δt and T=t .

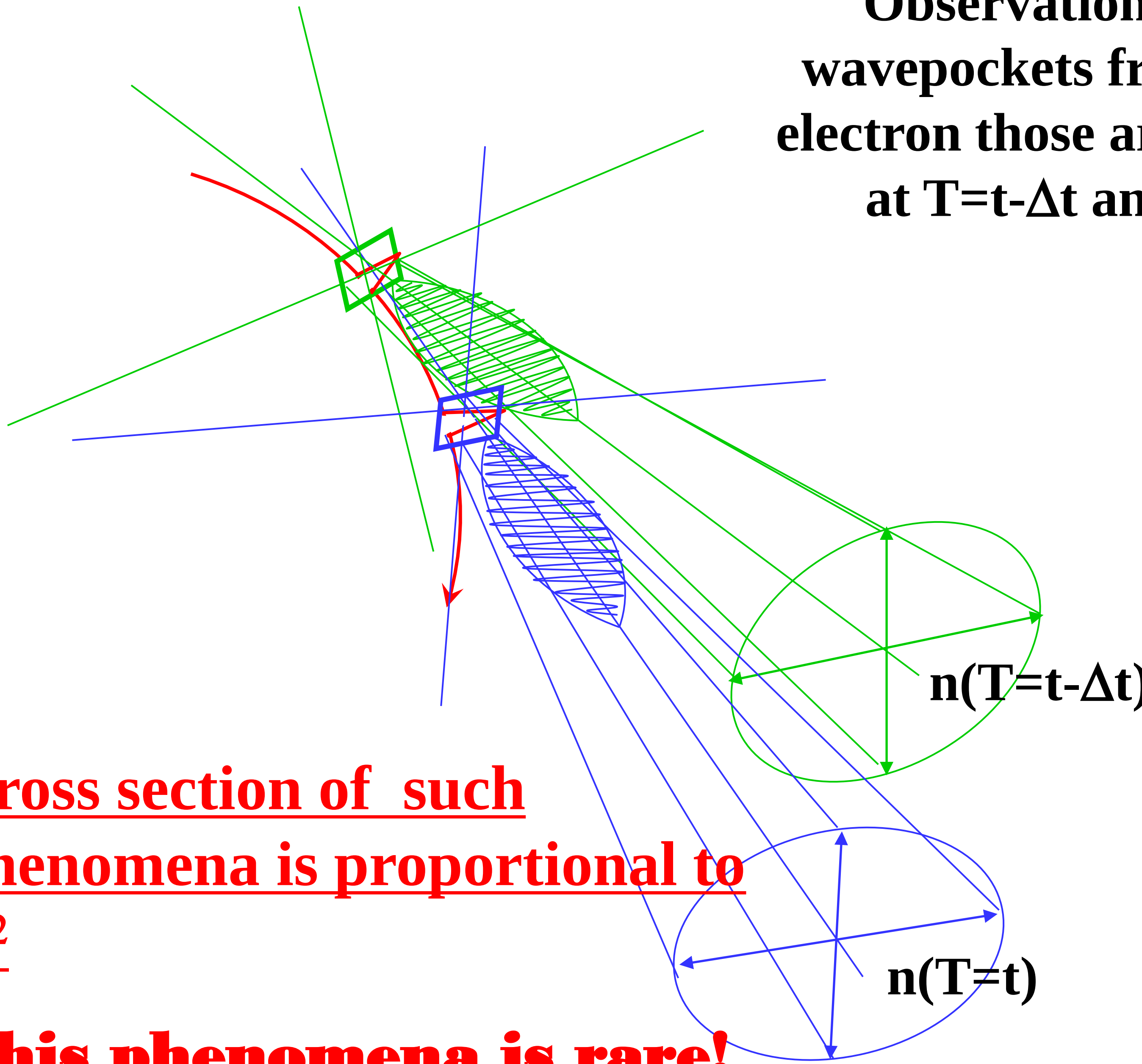


Cross section of such phenomena is proportional to α^2

$$\boldsymbol{E} = \boldsymbol{E}(t) + \boldsymbol{E}(t - \Delta t)$$

$$I = \left|\boldsymbol{E}(t) + \boldsymbol{E}(t - \Delta t)\right|^2$$

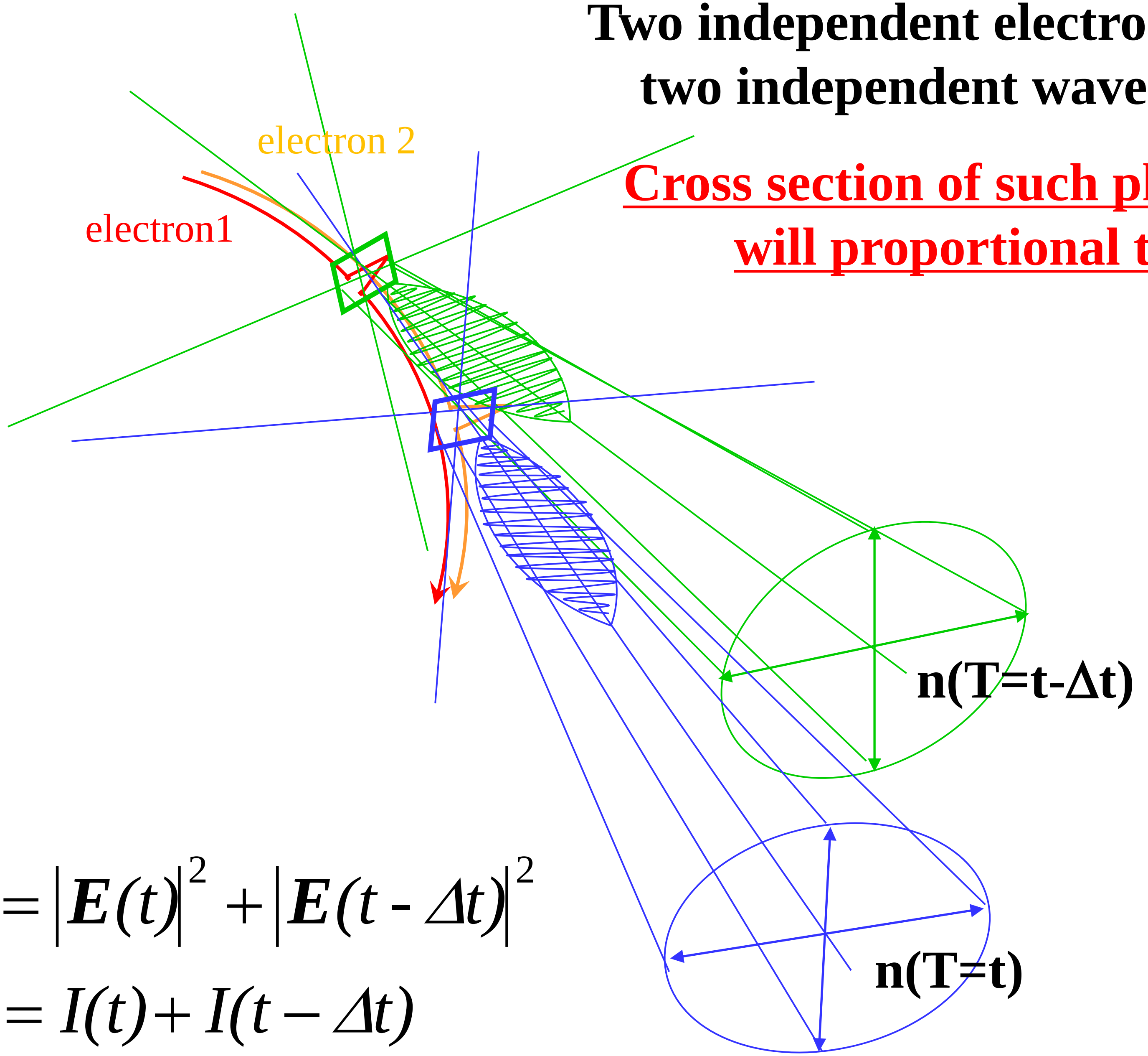
Observation of two wavepockets from single electron those are radiated at T=t-Δt and T=t .



<u>**Cross section of such phenomena is proportional to α^2**</u>

<u>**This phenomena is rare!**</u>

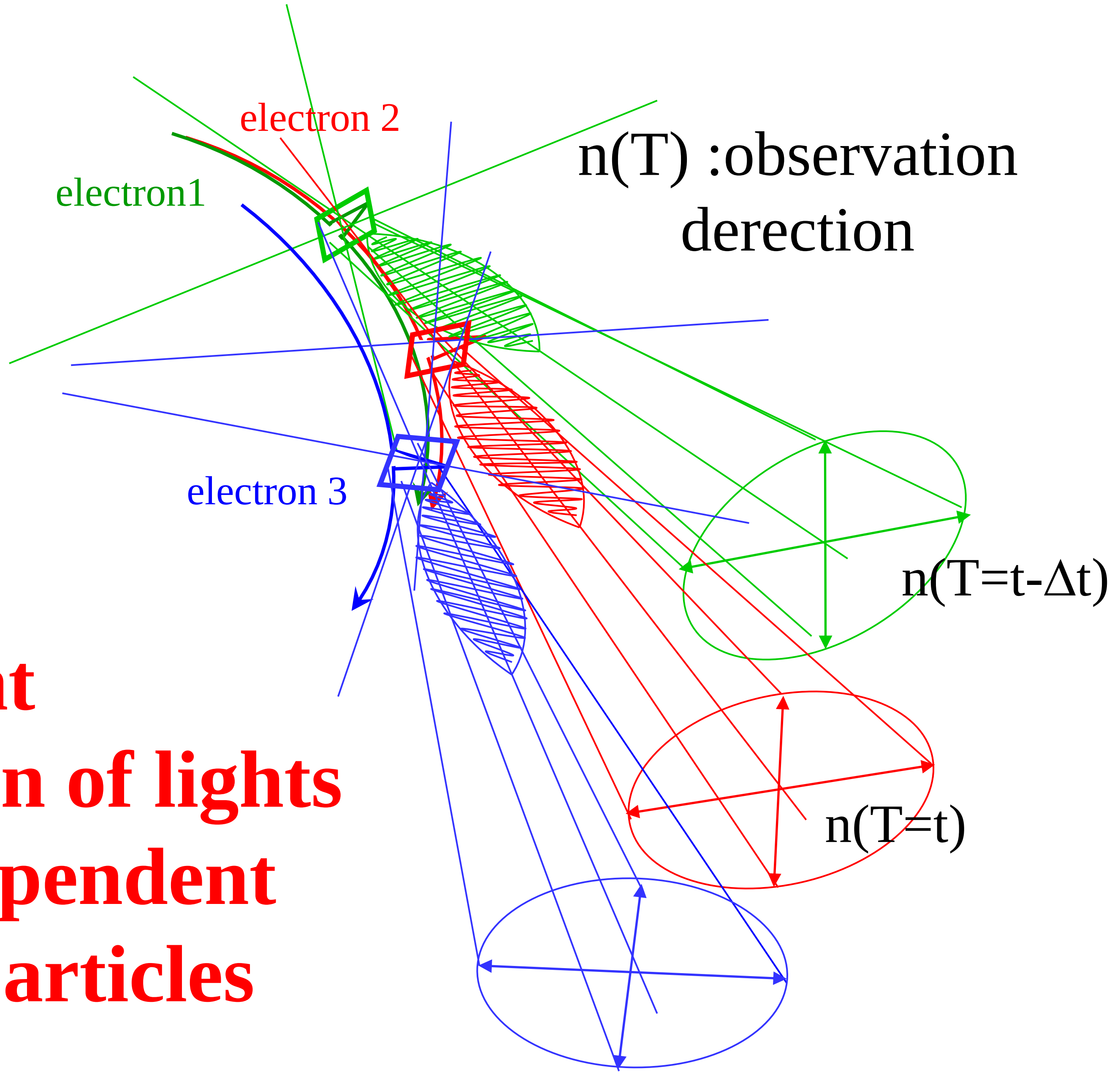
Two independent electrons irradiate two independent wave pockets .



Cross section of such phenomena will proportional to 2α.

$$I = \left|\boldsymbol{E}(t)\right|^2 + \left|\boldsymbol{E}(t - \Delta t)\right|^2$$

$$= I(t) + I(t - \Delta t)$$



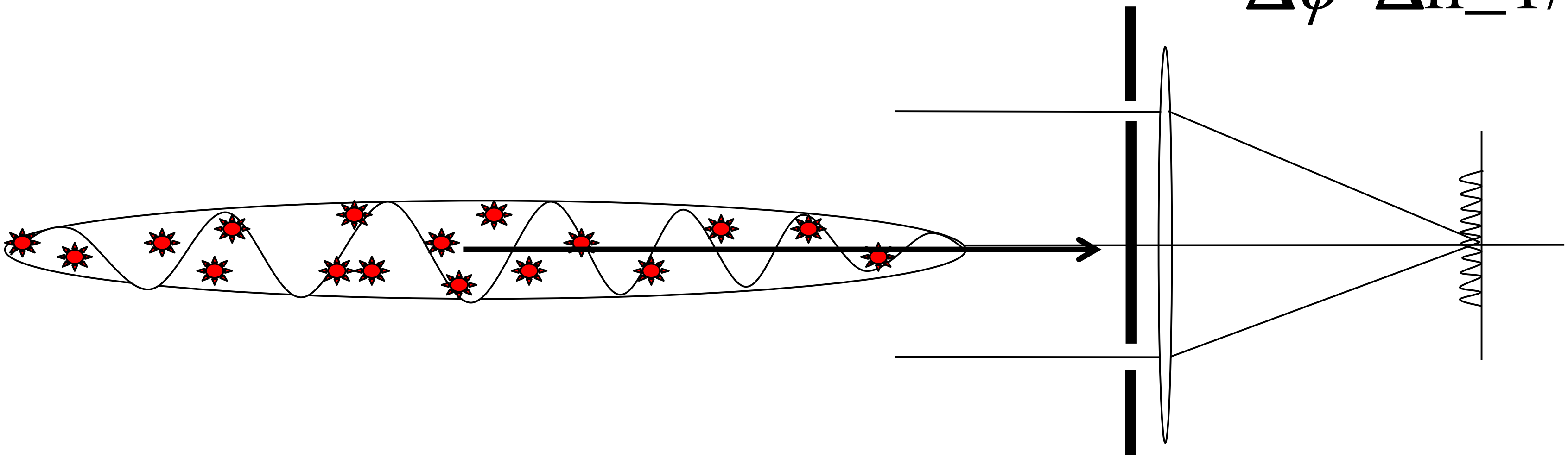
Incoherent summation of lights from independent charged particles

The understanding of "SR is incoherent summation of lights from independent electrons in shorter wavelength" is easily understand from <u>the intensity of SR is proportional to number of the electron, and not proportional to square of number of electrons.</u>

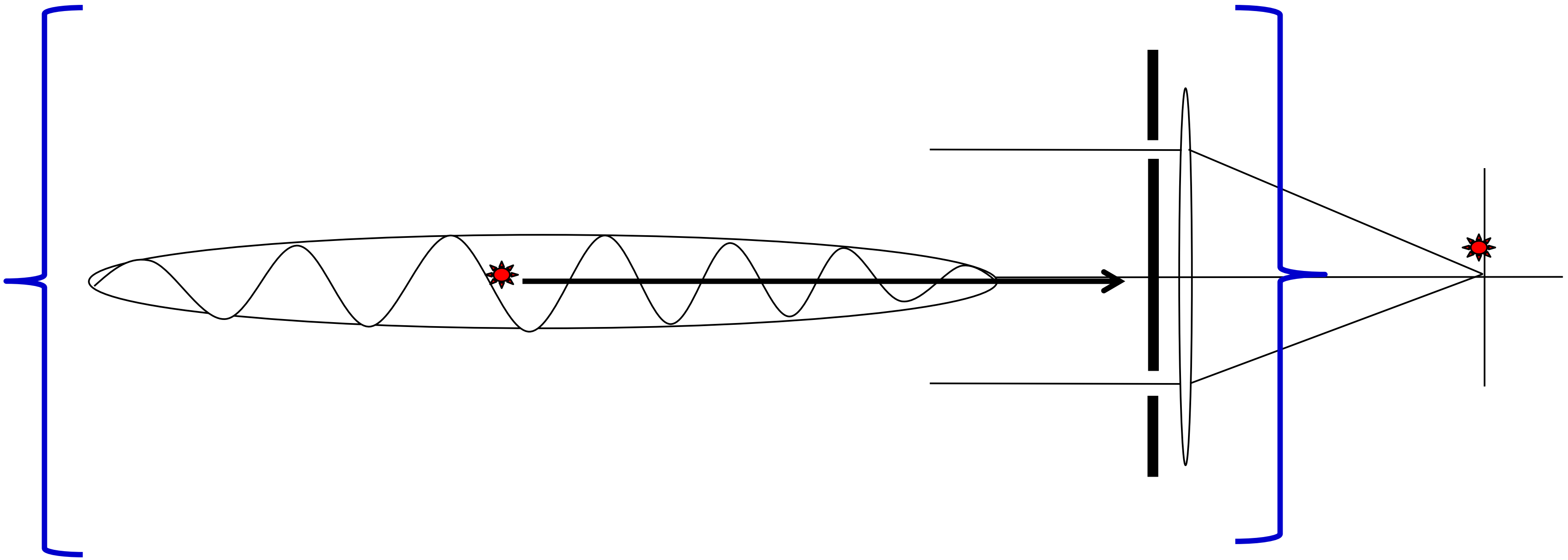
More clearly, the result of intensity interferometry indicate 1st order coherence between two photons are negligible, so "SR is incoherent summation of lights from independent electrons in shorter wavelength".

Again we need same discussion on n photons coherent state and 1 photon n times observation problem,
1 times observation of n photons in $|n\rangle$

$\Delta\phi \cdot \Delta n \geq 1/2$



n times observation of independent $|1\rangle$



More precisely, this problem is same as how to convert n photons in one wavepocket and one photon in n wavepockets.

We assume the most restrictive class of random process of photon statistics to the class of ergodic random process (long temporal average is equal large ensemble average). When n is large,

n times observation of one photon in the wavepocket

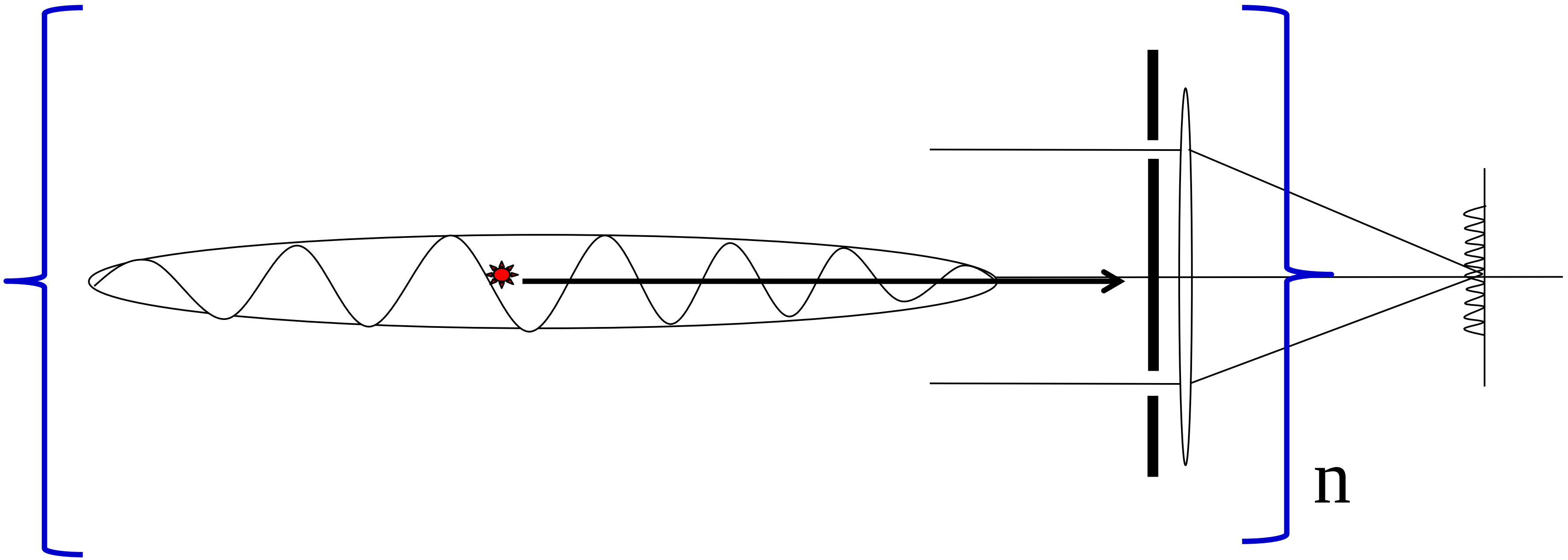
Temporal average

$\Updownarrow$

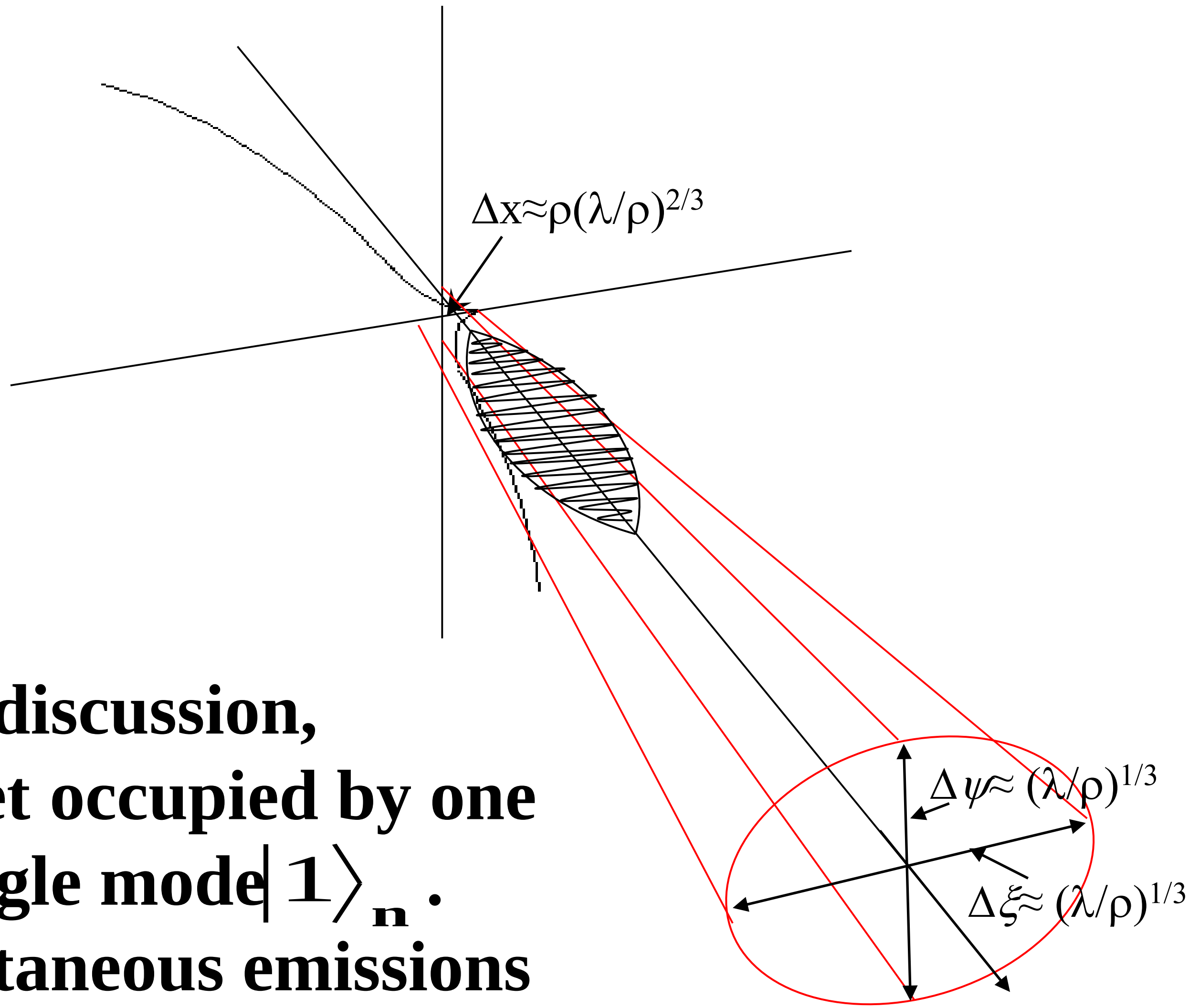
n photons are exited in the wavepocket.

Ensemble average

n times observation of independent $|1\rangle_{\mathbf{n}}$



$$\Delta\phi\cdot\Delta n \geq 1/2$$



From this discussion, wavepacket occupied by one photon single mode $|1\rangle_n$. And simultaneous emissions more then two photons are rare phenomena.

Using the expectation of the electric field E_1 of $|1\rangle_n$

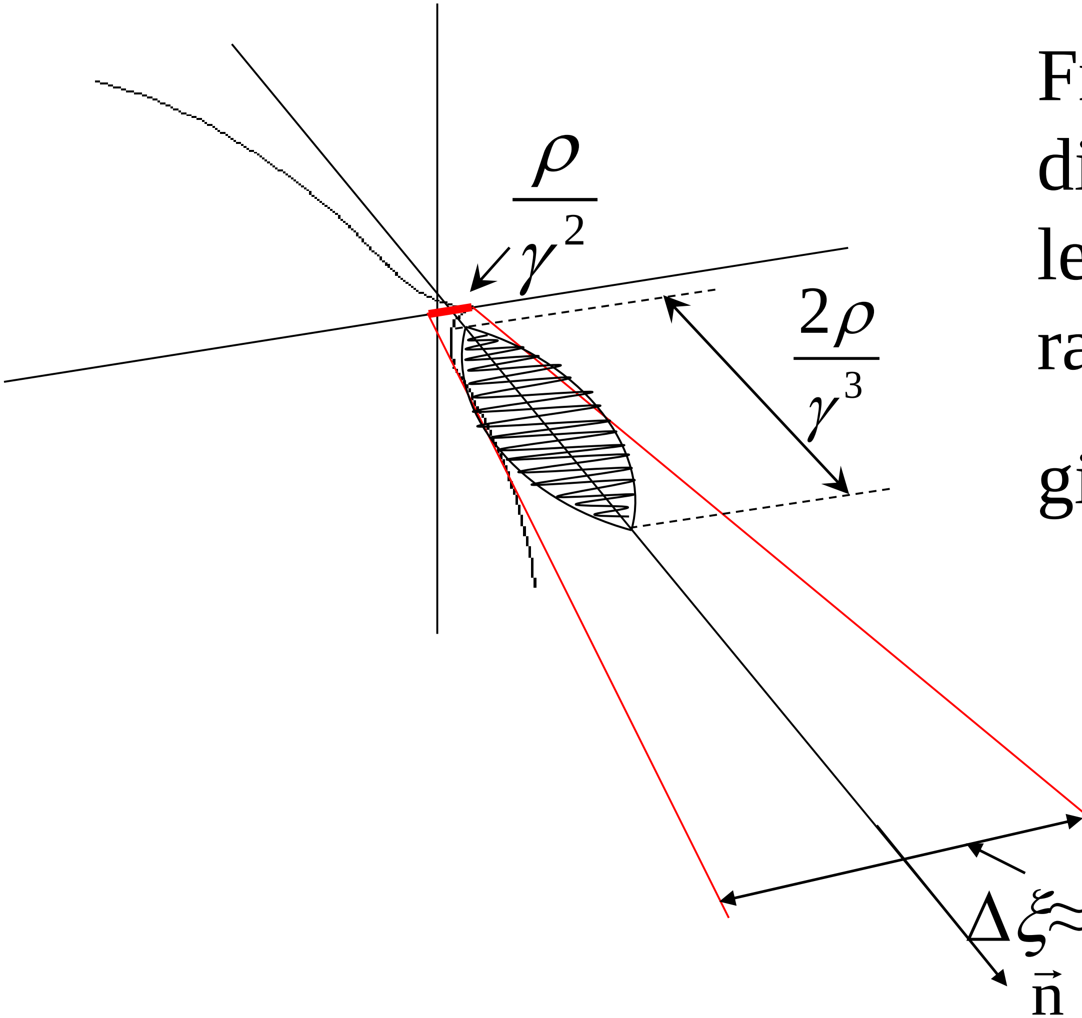
1st order temporal coherence is given by,

$$\left\langle E_1(t_1)E^*{}_1(t_2)\right\rangle = \int E_1(x,t)E^*1(x,t+t')dt$$

For the 1st order spatial coherence, applying the Van Cittert-Zernike's theorem to the 1st order temporal incoherent lights from ensemble of independent electrons in the bunch, it is given by Fourier Transform of transverse bunch shape taking into account of incoherent field depth. More detail discussion for 1st order spatial coherence, please see interferometer measurement in later chapter.

Note independent photons emitted from independent charged particles has no 1st order coherence!

Coherent length (or coherent field depth) of the synchrotron radiation. Coherent length is just length of wavepacket, and it is given by Fourier transform of the spectrum.



From the previous discussion, coherent length of synchrotron radiation at λ_c is given by $\frac{2\rho}{\gamma^3}$.

Discussion for field depth

Note, this coherent length (coherent field depth) has not a same meaning as field depth in diffraction theory.

In the diffraction theory, we assumed input field of light is quasi-monochromatic (long coherent length).

The field depth in focusing system represented by diffraction theory is defined from longitudinal diffraction.

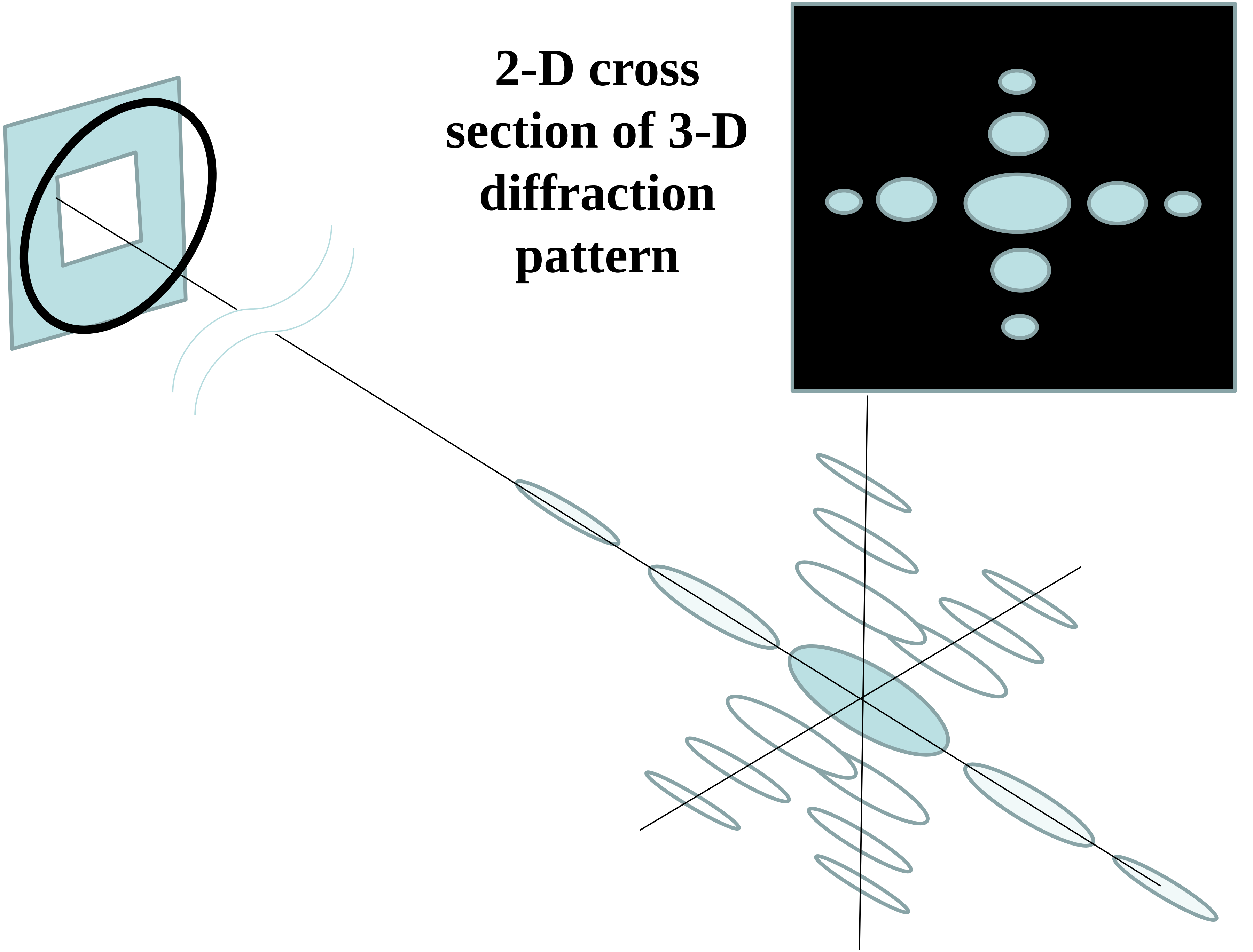
Increases the coherence length of the wavepacket by monochrometor such as band-pass filter

For 633nm visible light and a 1nm bandpass filter

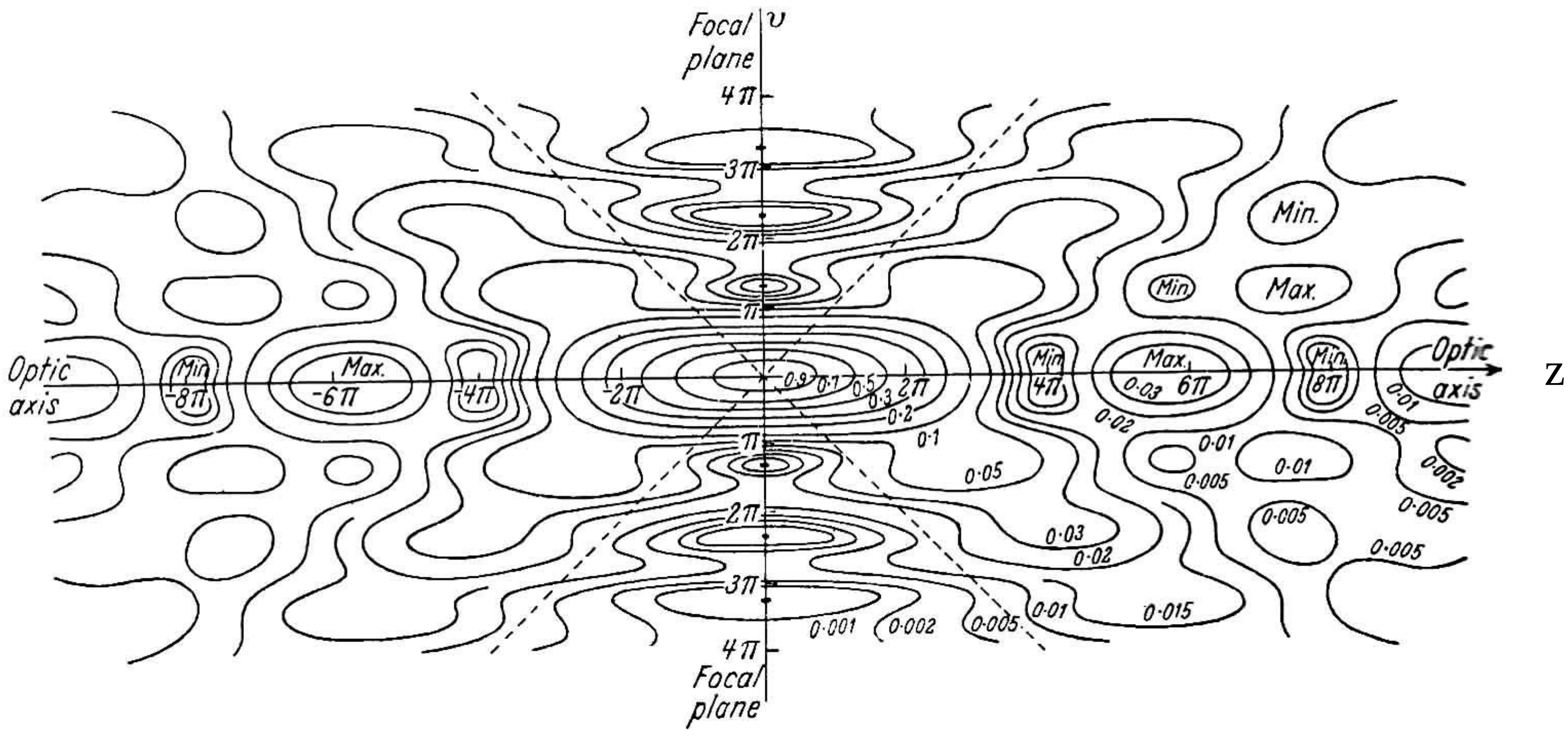
$$\delta t = \lambda^2 / c\delta\lambda = \frac{\left(633\times10^{-9}\right)^2}{\left(3\times10^{8}\right)\left(1\times10^{-9}\right)} = 1.3ps$$

$$= 0.39mm$$

2-D cross section of 3-D diffraction pattern



Cross section of longitudinal diffraction pattern



Intensity of longitudinal on axis

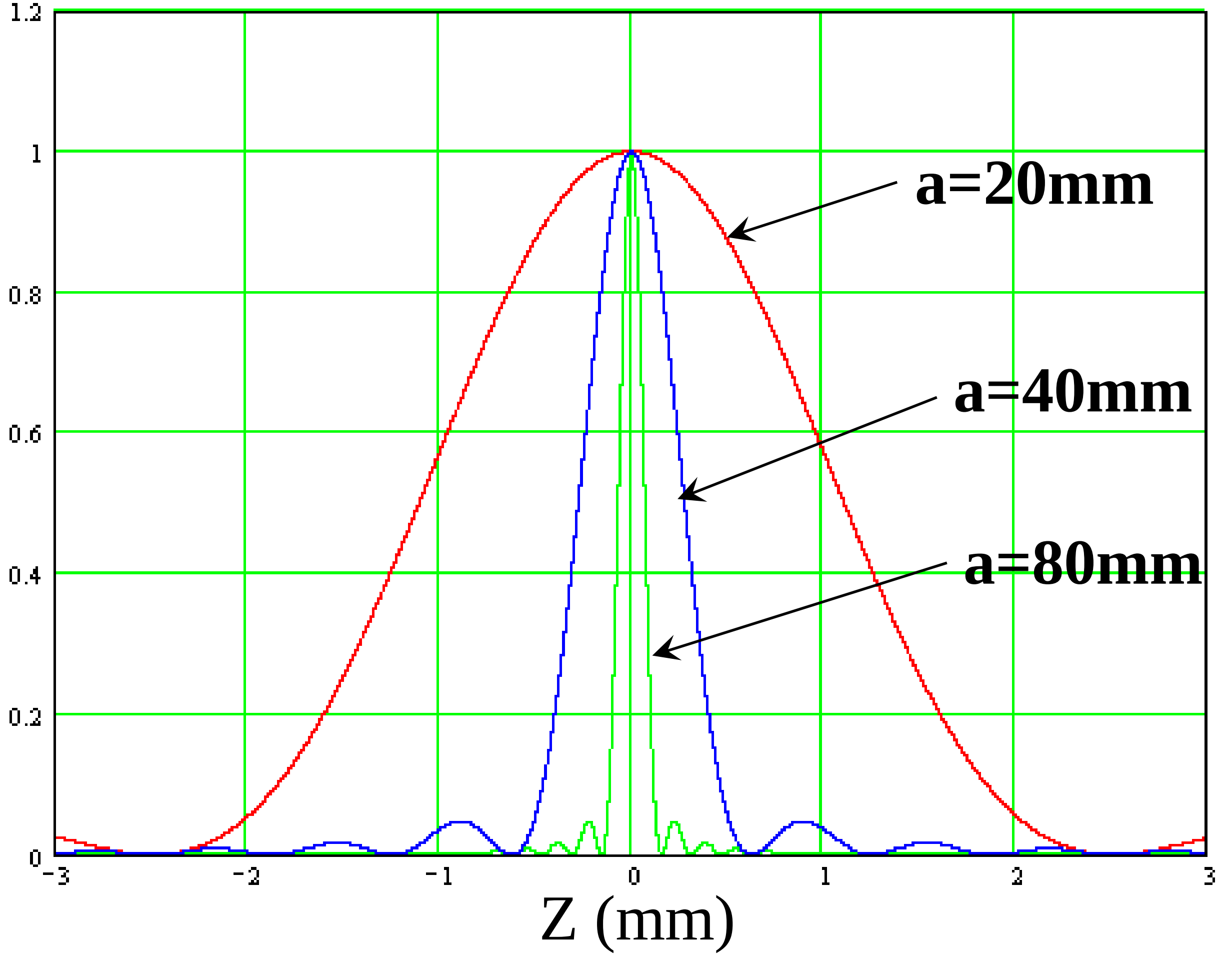
$$N = \frac{a^2}{\lambda \cdot f}\ , \qquad U(z) = \frac{2\pi}{\lambda}\left(\frac{a}{f}\right)^2 z$$

$$V(z) = \frac{U(z)}{1 + \frac{U(z)}{2\pi\, N}}$$

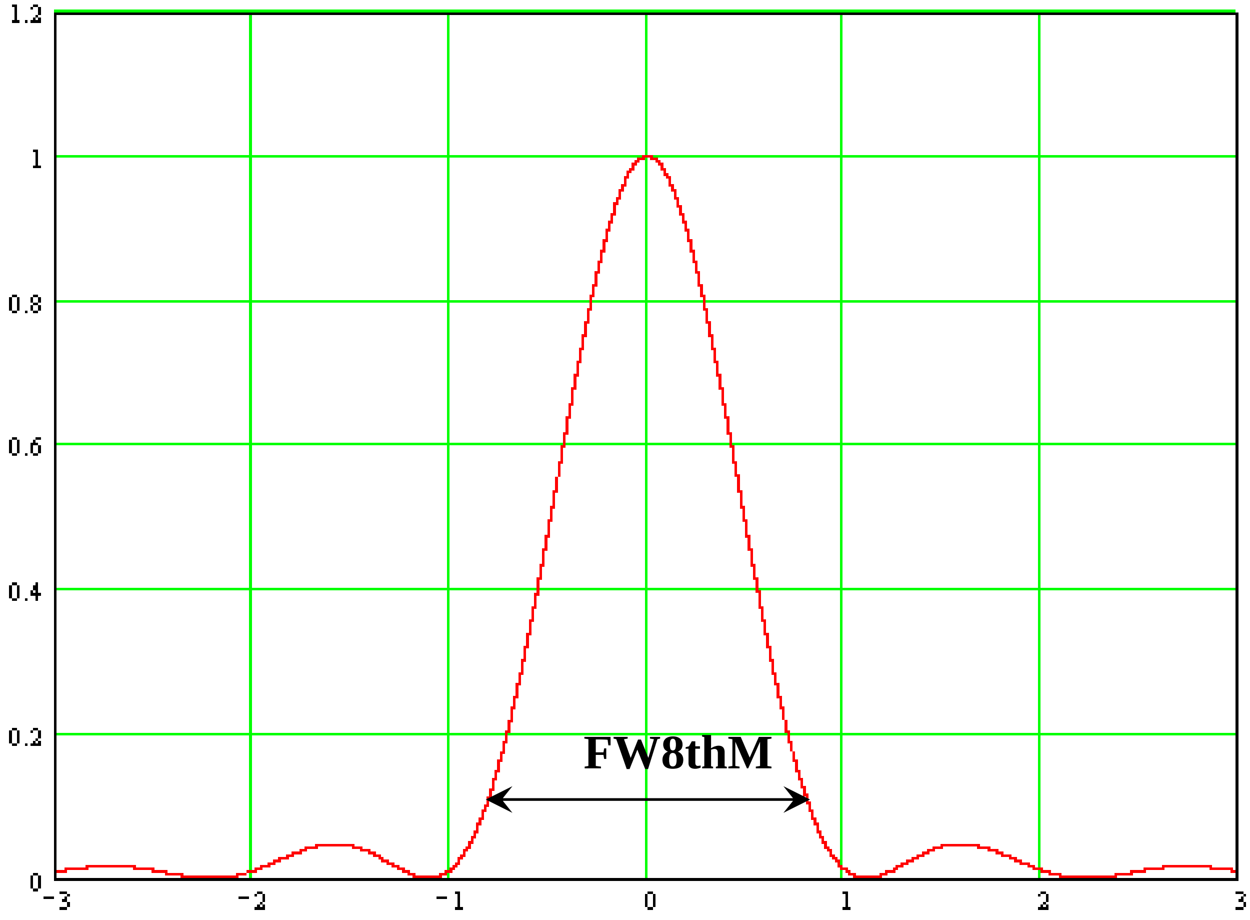
$$I(z) = \left(1 - \frac{V(z)}{2\pi\, N}\right)\left(\frac{\sin\left(\frac{V(z)}{4}\right)}{\frac{V(z)}{4}}\right)^2$$

Longitudinal diffraction pattern for lens f=1000mm,

a=80, 40, 20mm



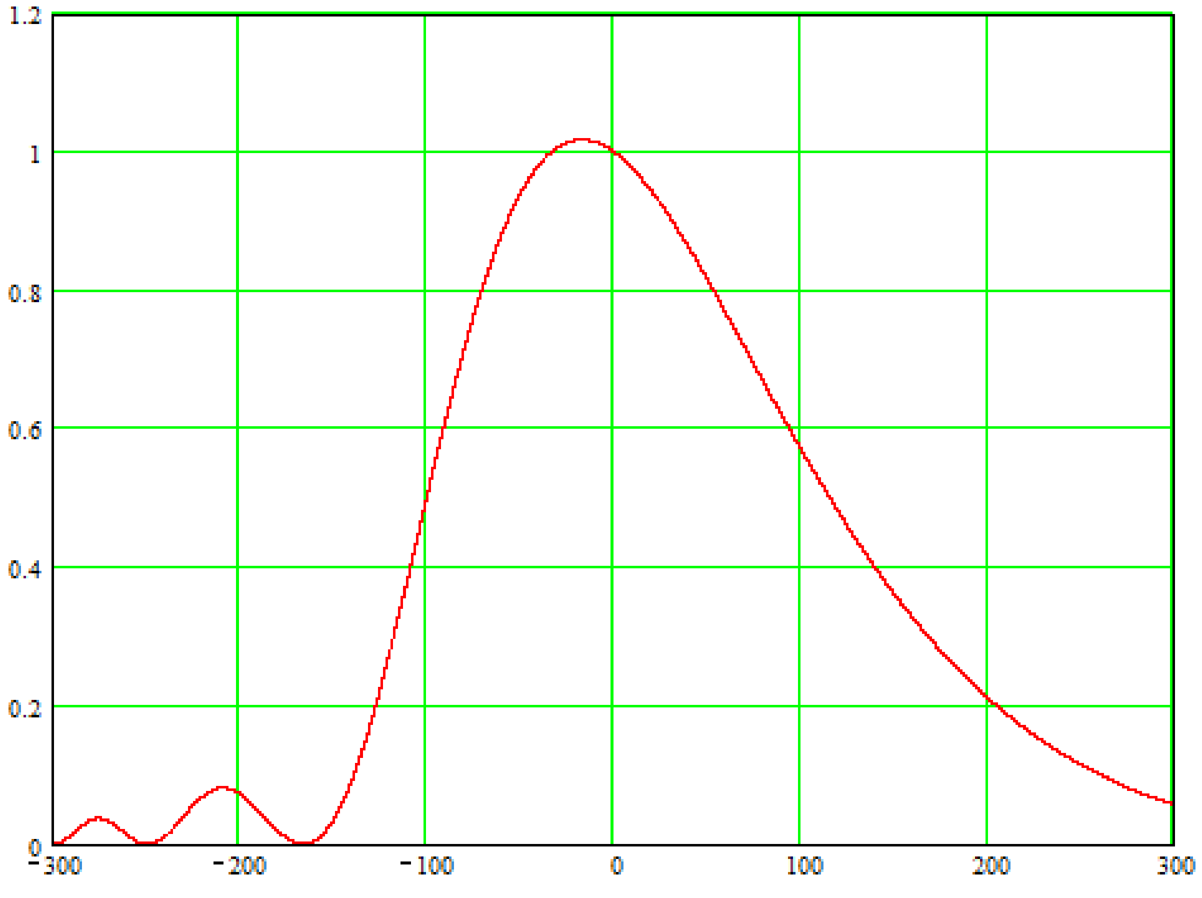
Definition of field depth=Full Width at 8th Maximum



Z (mm)

Interferometer optics a=1mm, f=500mm

Note, longitudinal diffraction pattern has asymmetry for small aperture.



Z (mm)

Return to corresponding source point, we need scale with longitudinal magnification.

Let denote transverse magnification by β, the longitudinal magnification η is given by ,

$$\eta = \beta^2$$

For example, in the case of

Source point to objective lens: 25000mm

Focus length of objective lens 500mm

$\beta = 0.0204$

$\eta = 0.0004165$

So, $\pm$100mm diffraction field depth will cover

$\pm$ 2400mm at the source point.

Filed depth(defined by diffraction)

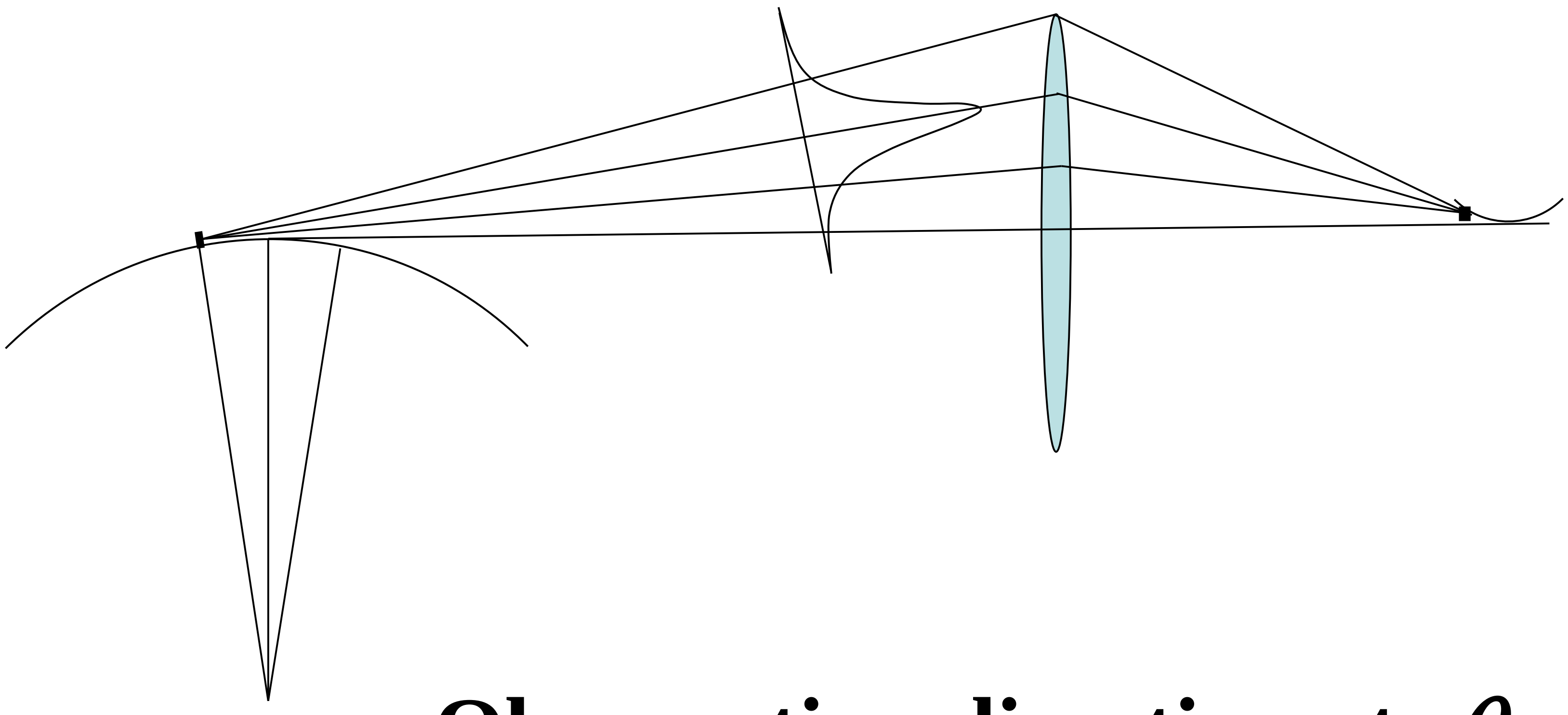
$\neq$

coherent length of light

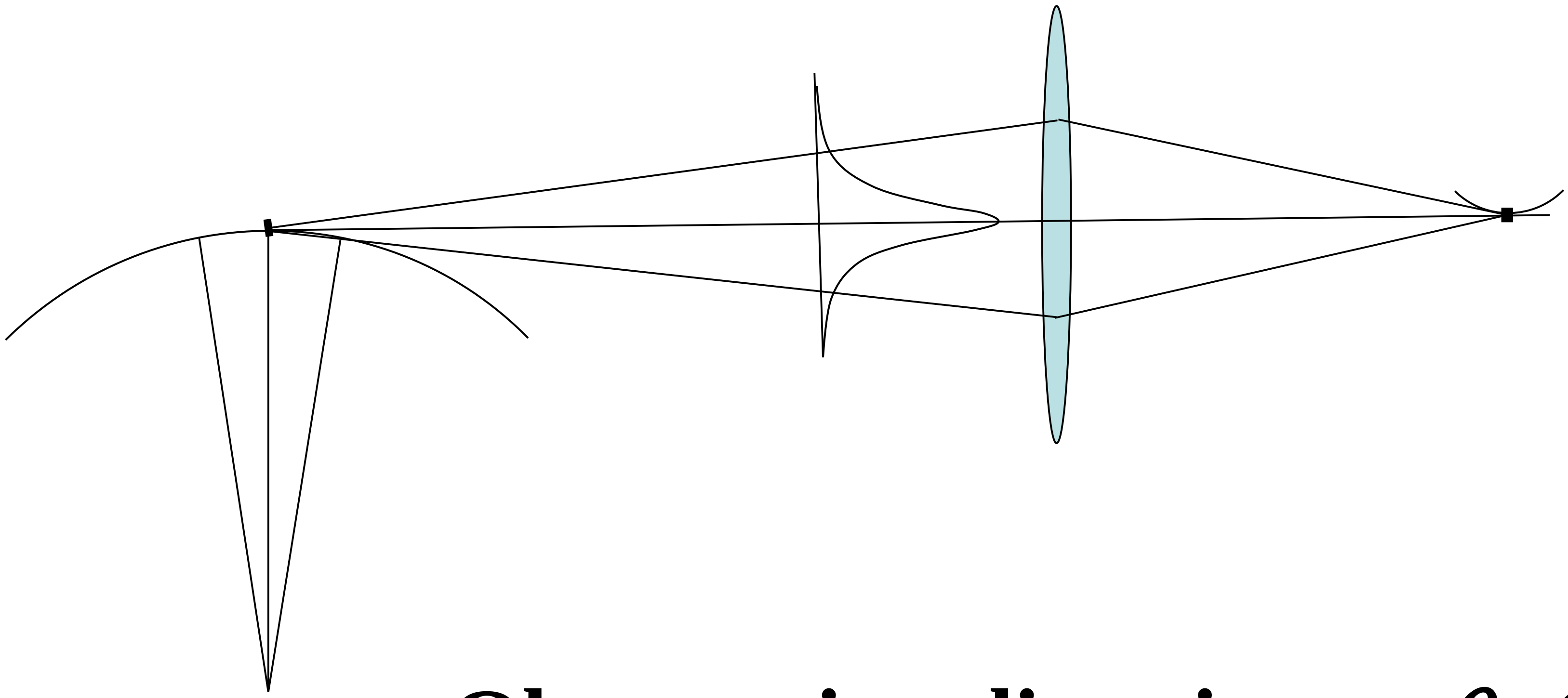
(length of wave pocket or coherent field depth)

Actually, ensemble of incoherent lights come from source point makes Field depth as the longitudinal region in focus.

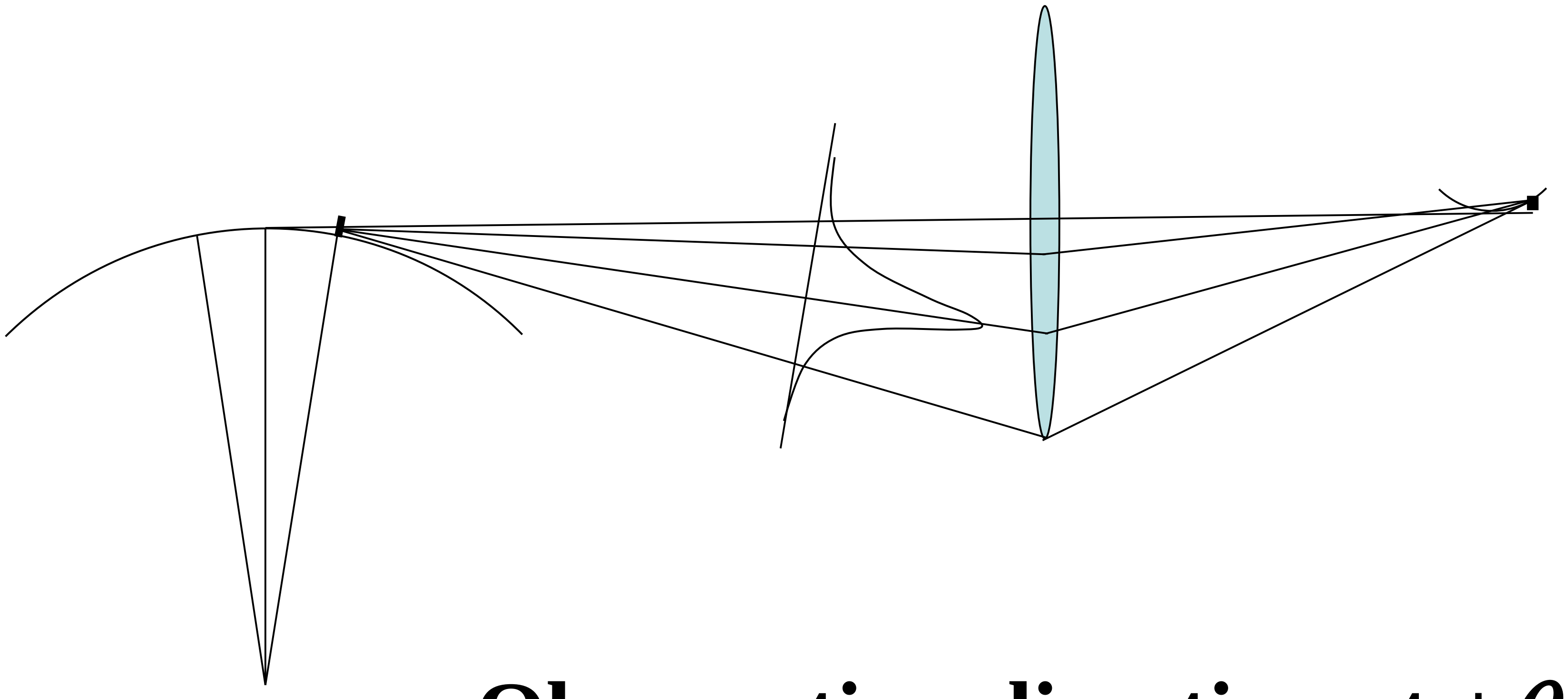
In generally, from this view point Field depth should say as Incoherent field depth .



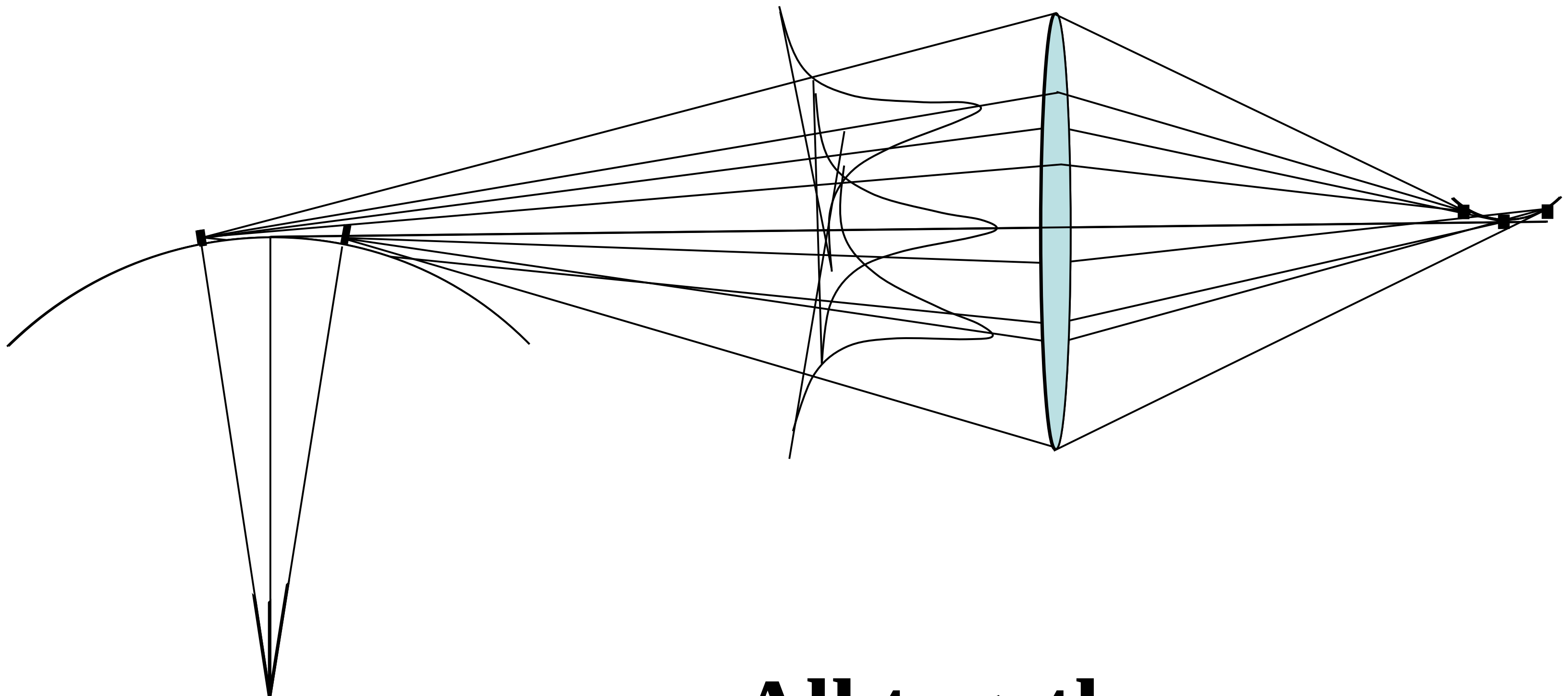
Observation direction at -θ



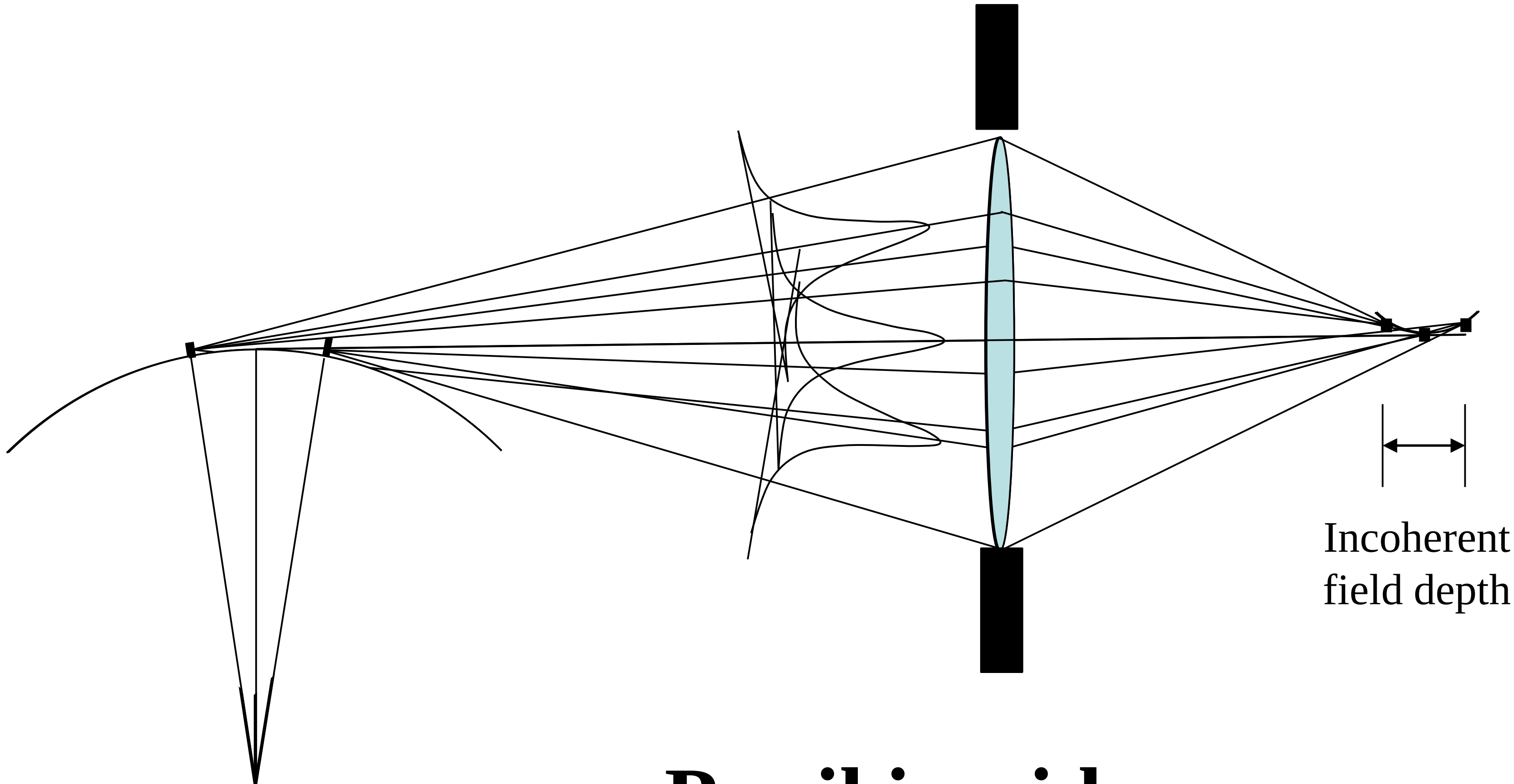
Observation direction at $\theta=0$



Observation direction at +θ

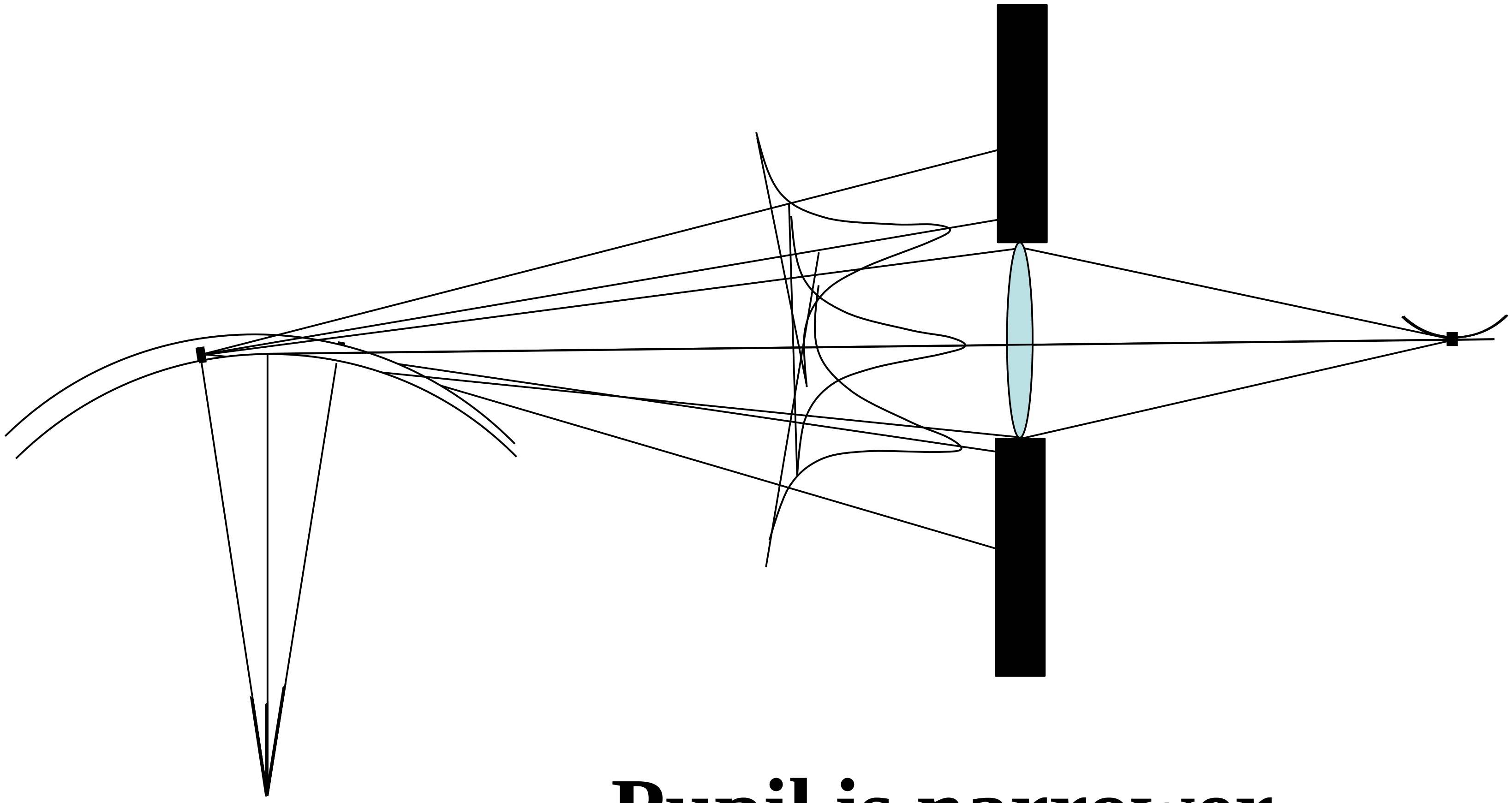


All together



Pupil is wider

PSF on the focus point is limited by opening of SR



Pupil is narrower

PSF on the focus point becomes opening of pupil

When the incoherent field depth <diffraction field depth

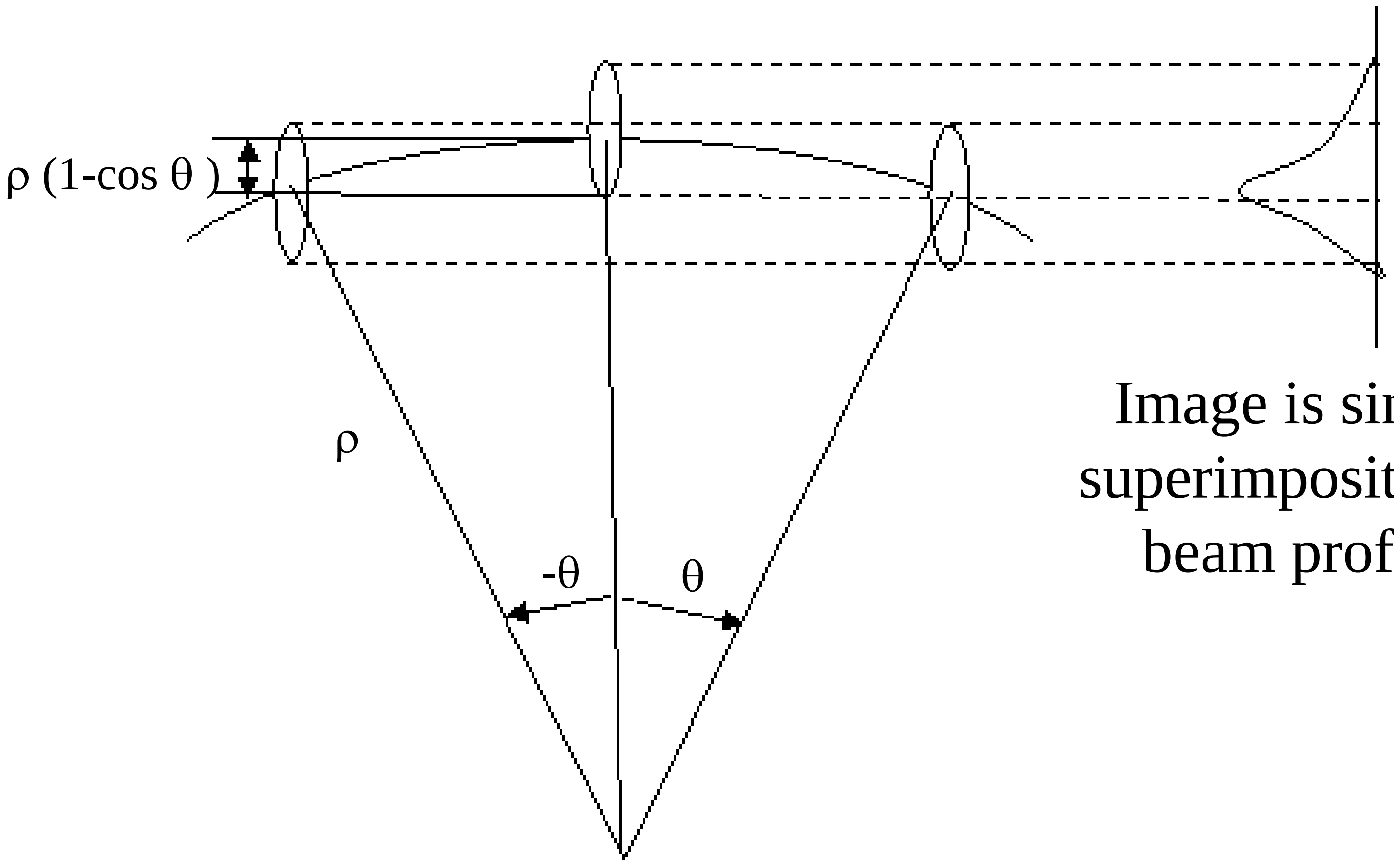
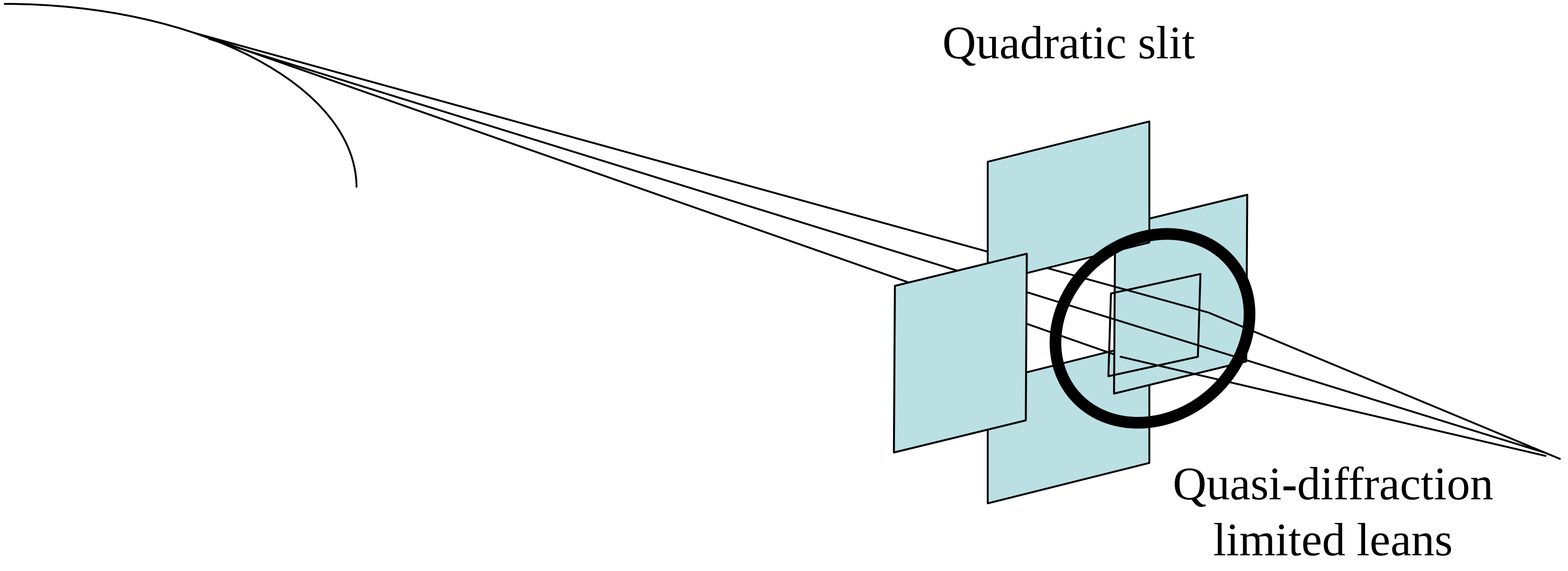
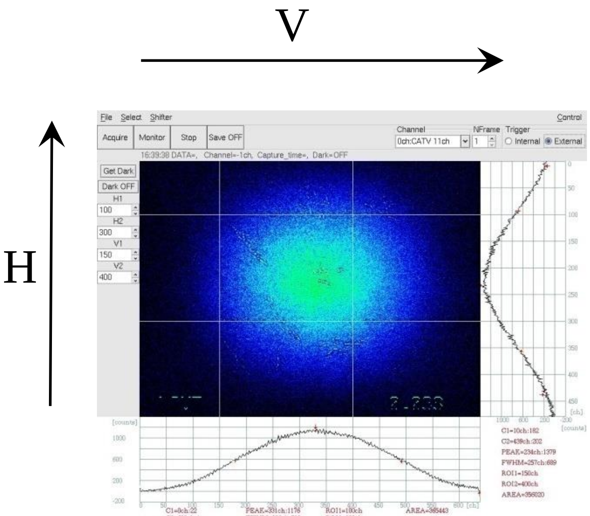


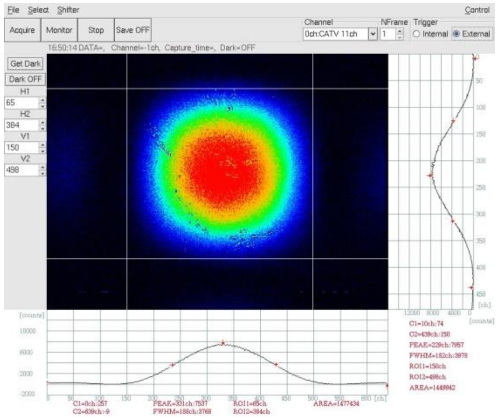
Image is simply superimposition of beam profiles

Observation of incoherent field depth with quasi-diffraction limited focusing system at the ATF

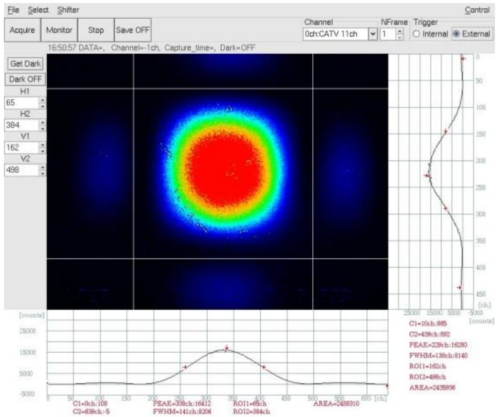




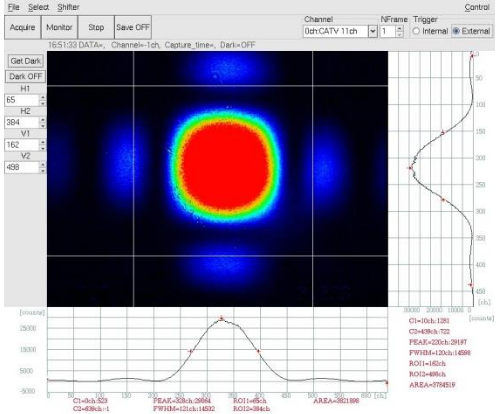
2mmx2mm



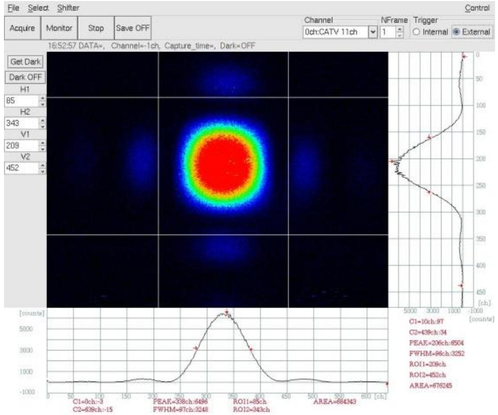
3x3



4x4



5x5



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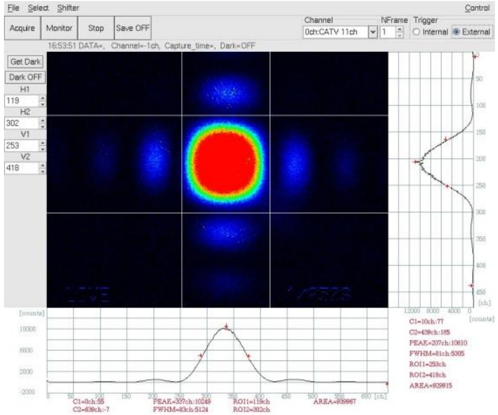

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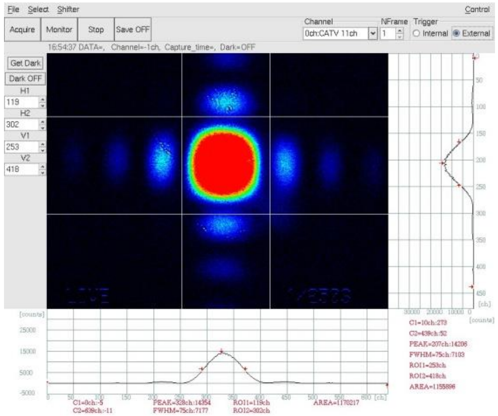

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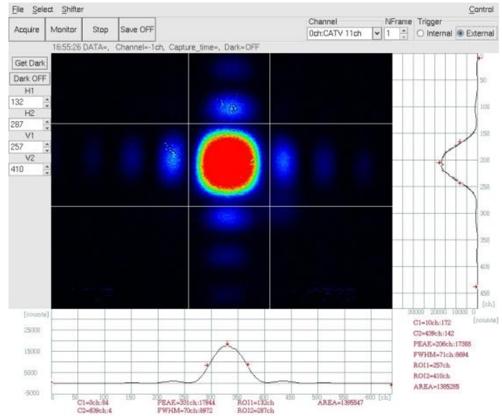

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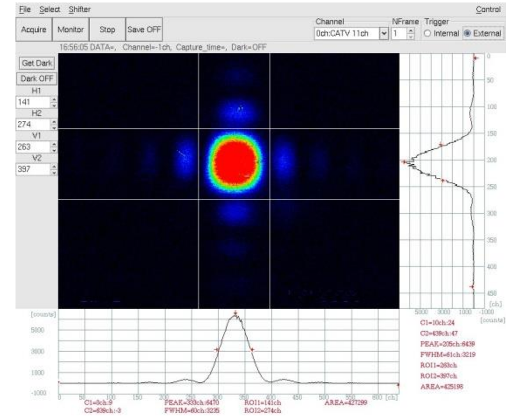

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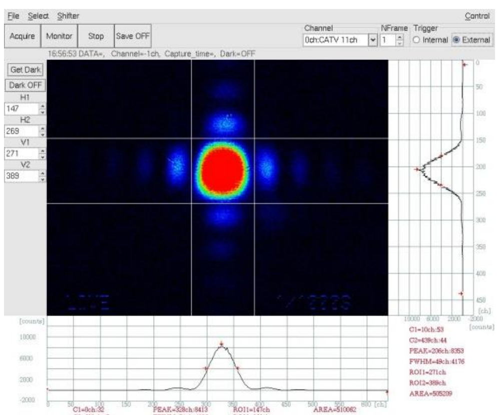
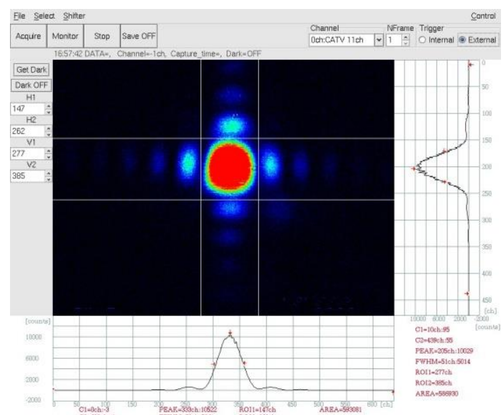


11x11



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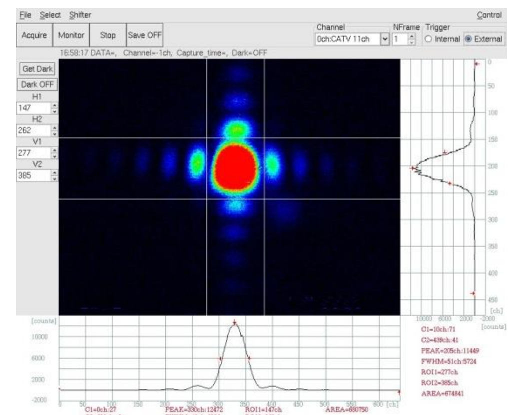

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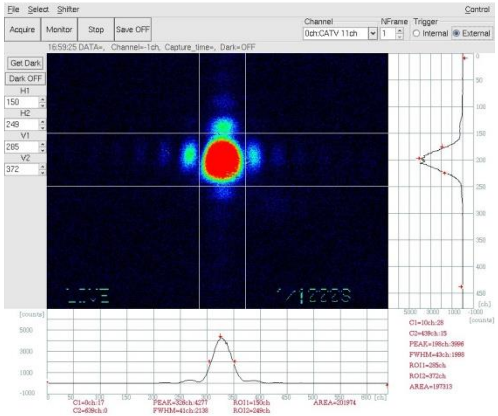

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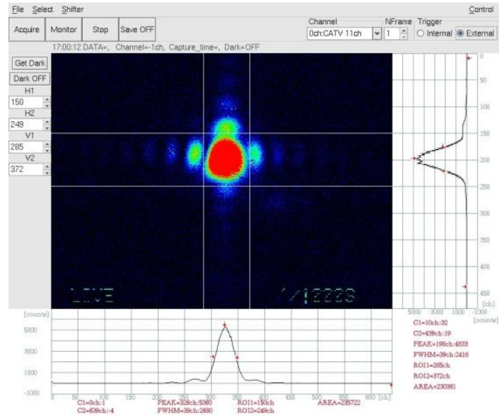

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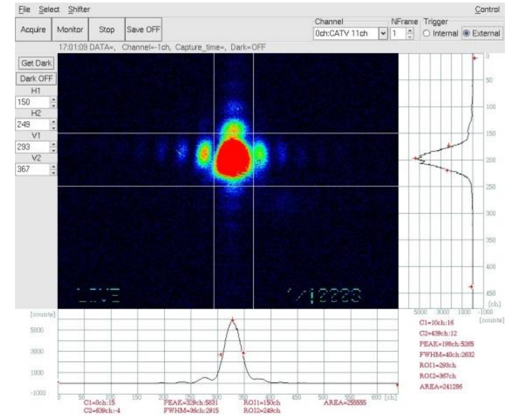

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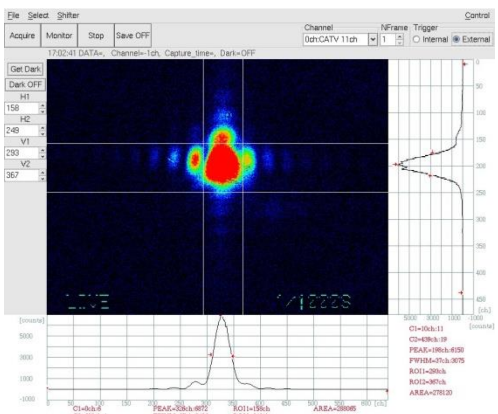

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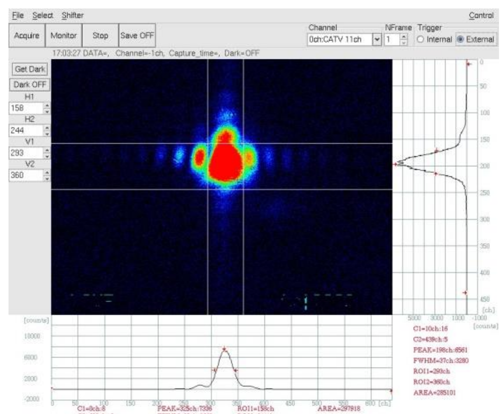

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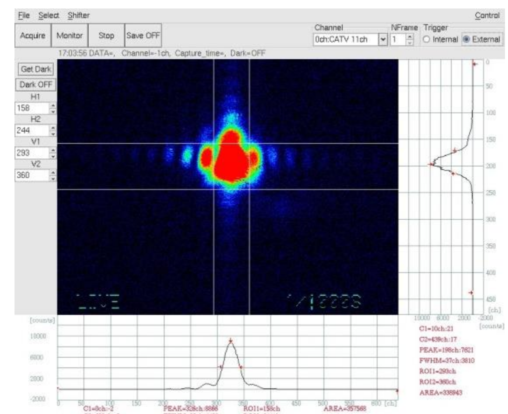

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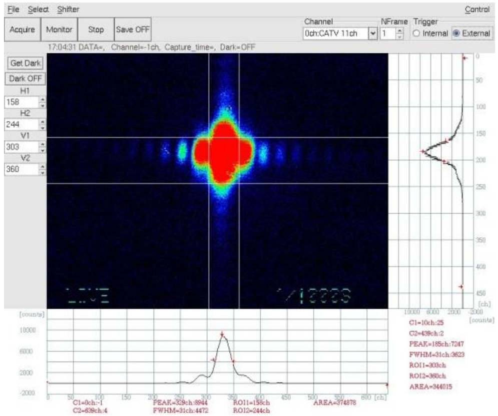

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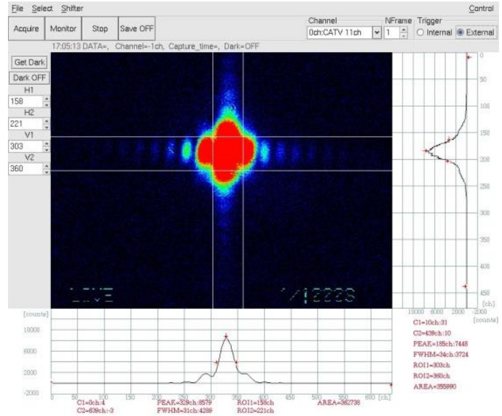

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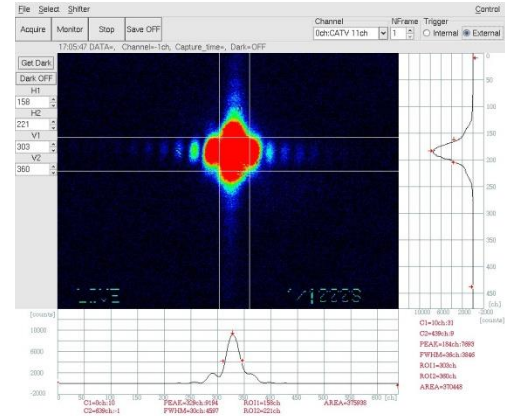

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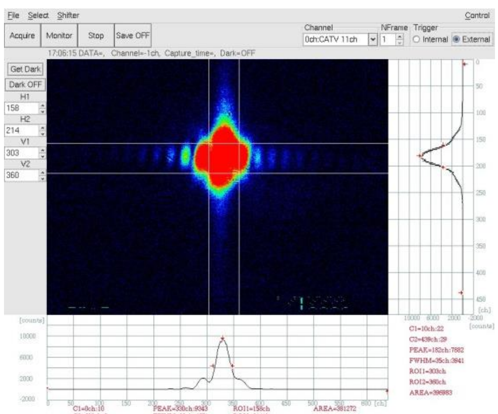
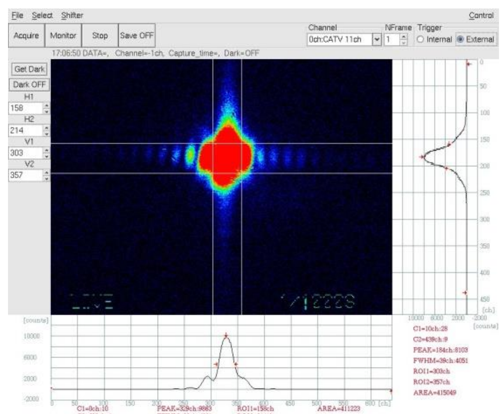
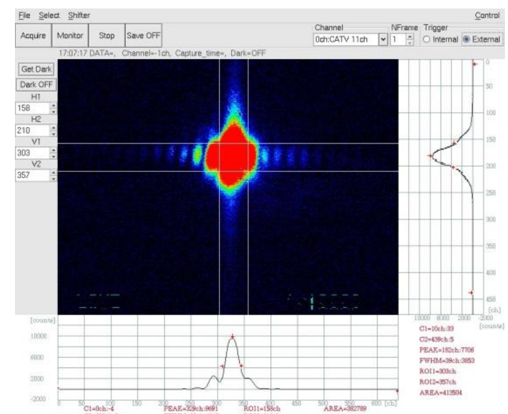


24x24



25x25



26x26

Lecture 6

SR interferometer

3.1 Theoretical background of interferometry

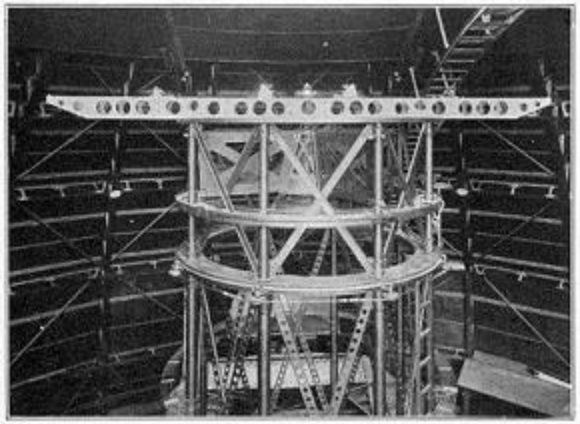
To measure a size of object by means of spatial coherence of light (interferometry) was first proposed by H. Fizeau in 1868!

This method was realized by A.A. Michelson as the measurement of apparent diameter of star with his stellar interferometer in 1921.

This principle was now known as " Van Cittert-Zernike theorem" because of their works;

1934 Van Cittert

1938 Zernike.



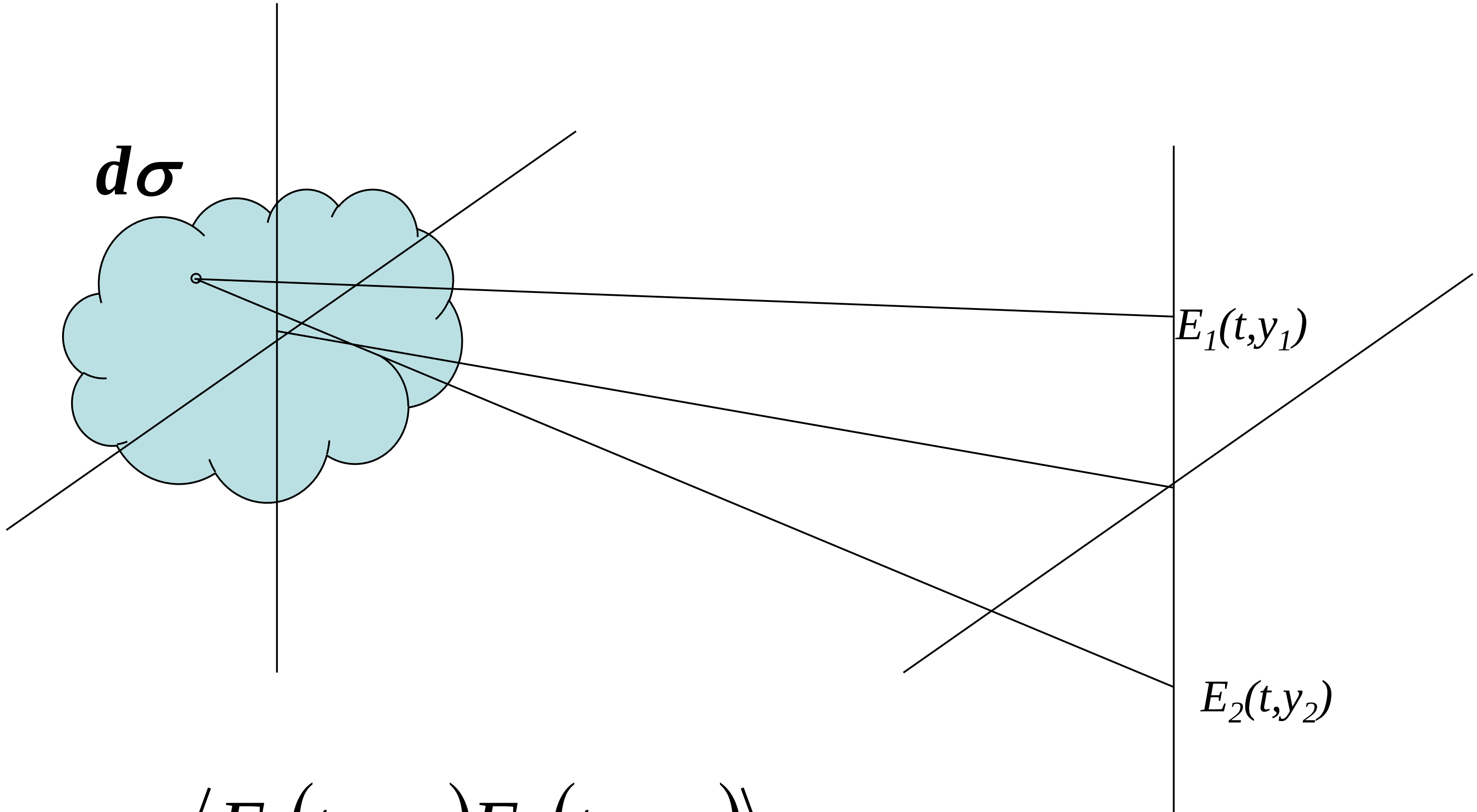
Spatial coherence and profile of the object
Van Cittert-Zernike theorem

According to van Cittert-Zernike theorem, with the condition of light is temporal incoherent (no phase correlation), the complex degree of spatial coherence $\gamma(\upsilon_x,\upsilon_y)$ is given by the Fourier Transform of the spatial profile f(x,y) of the object (beam) at longer wavelengths such as visible light.

$$\gamma(\upsilon_x, \upsilon_y) = \int\int f(x, y) \exp\{-i \cdot 2 \cdot \pi(\upsilon_x \cdot x + \upsilon_y \cdot y)\} dxdy$$

where υ_x,υ_y are spatial frequencies given by;

$$\upsilon_x = \frac{D_x}{\lambda \cdot R_0} \ , \quad \upsilon_y = \frac{D_y}{\lambda \cdot R_0}$$



$$g_{12} = \langle E_1(t\ , y_1) E_2(t\ , y_2) \rangle$$

$$E_j(t\ , y_j) = a(t\ , y_j)\frac{1}{r_j}\exp\{-2\pi\, i(\omega t - k y_j)\}$$

$$g_{12} = \left\langle E_1(t\,,y_1)E_2(t\,,y_2)\right\rangle$$
$$= \frac{\left\langle a(t\,,y_1)\,a^*(t\,,y_2)\right\rangle}{r_1 \cdot r_2}\exp\{-2\pi\, k(y_1 - y_2)\}$$
$$= I(\sigma)d\sigma\,\frac{1}{r_1 \cdot r_2}\exp\{-2\pi\, k(y_1 - y_2)\}$$

Then first order mutual coherence by whole area is given by integrate $\mathbf{g_{12}}$;

$$\boldsymbol{G}_{12}(y_1, y_2) = \int_{\Sigma} I(\sigma)\frac{1}{r_1 \cdot r_2}\exp\{-2\pi\, k(y_1 - y_2)\}\,d\sigma$$

$$\gamma_{12}(y_1, y_2) = \frac{G_{12}}{G_{11}G_{22}}$$

$$= \frac{\int_{\Sigma} I(\sigma)\exp\{-2\pi\, i\, k(y_1 - y_2)\}\, d\sigma}{\int_{\Sigma} I(\sigma)\ d\sigma}$$

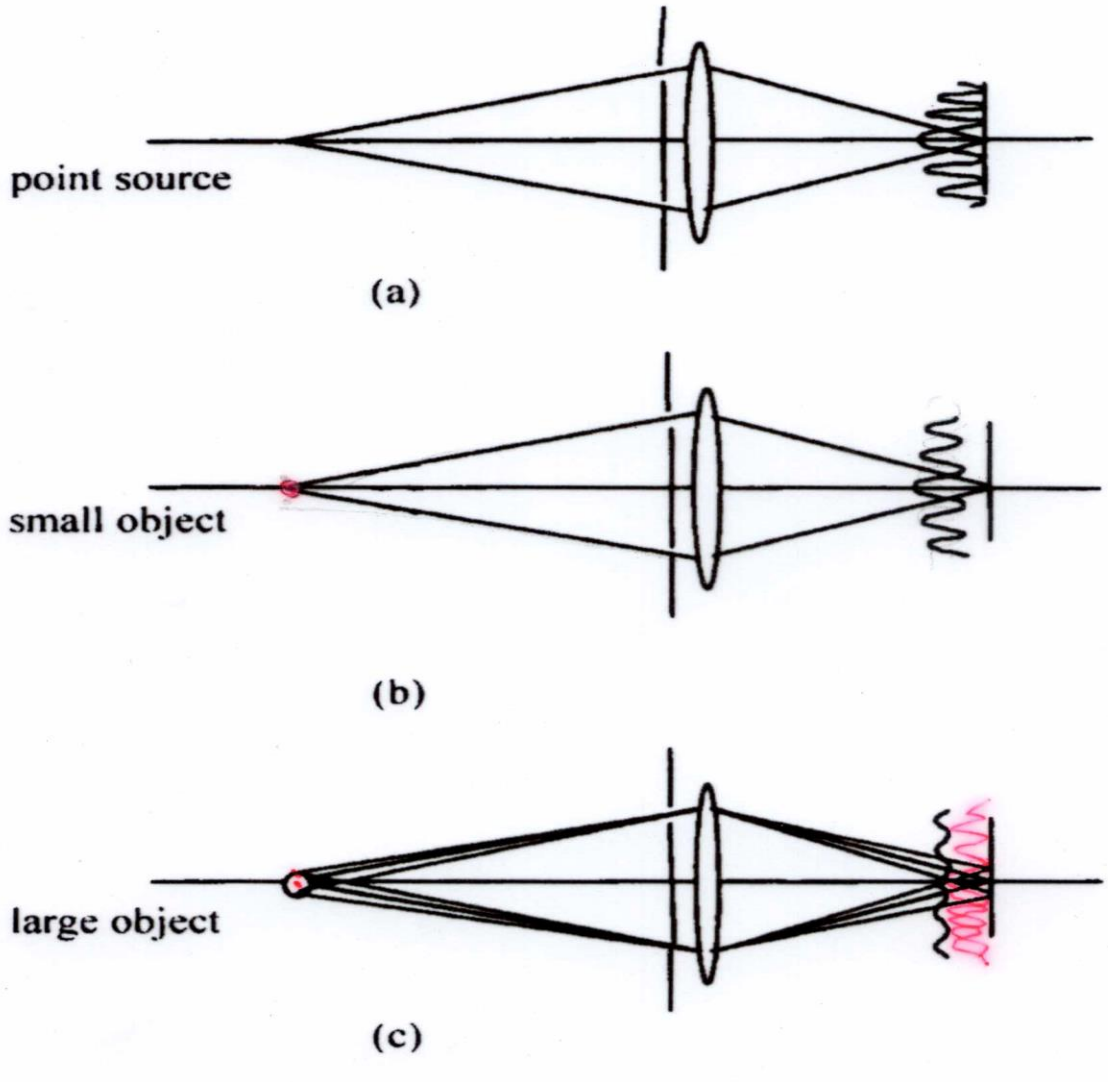
Introducing normalized intensity distribution $\mathbf{I}$,

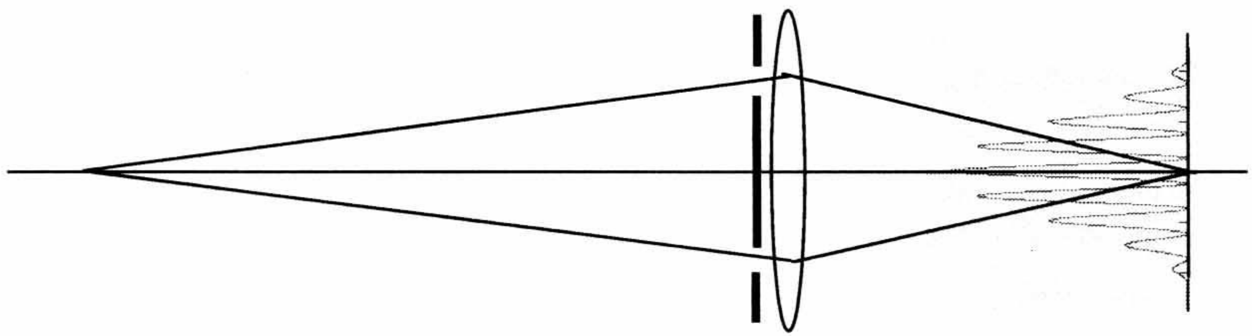
$$\gamma_{12}(y_1, y_2) = \int_{\Sigma} \mathbf{I}(\sigma)\exp\{-2\pi\, i\, k(y_1 - y_2)\}\, d\sigma$$

Representing $\mathbf{I}$ **as 2D distribution, and using spatial frequency ;**

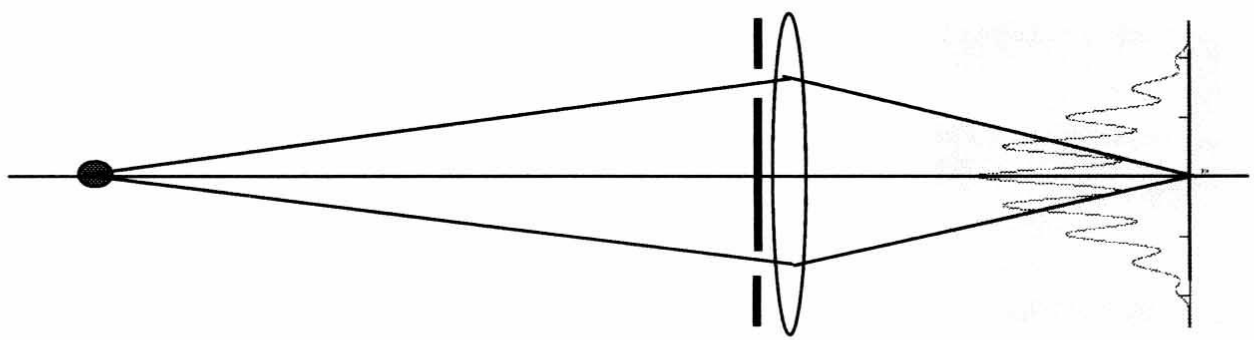
$$\gamma_{12} = \iint \mathbf{I}(x_o, y_o)\exp\{-2\pi\, i\left(\upsilon_x\ x_o + \upsilon_y\ y_o\right)\} d\sigma$$

Simple understanding of van Cittert-Zernike theorem



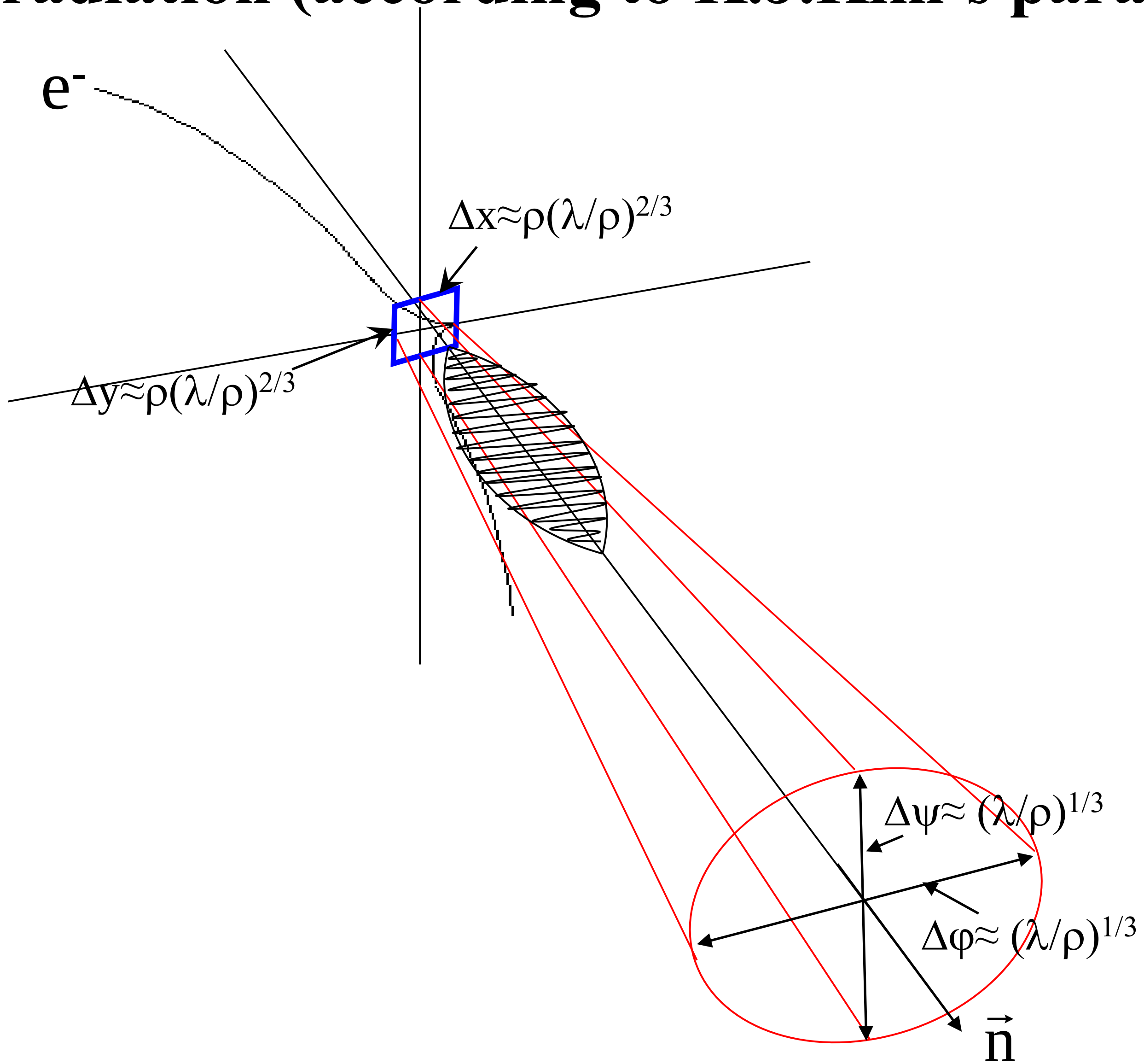


(a)

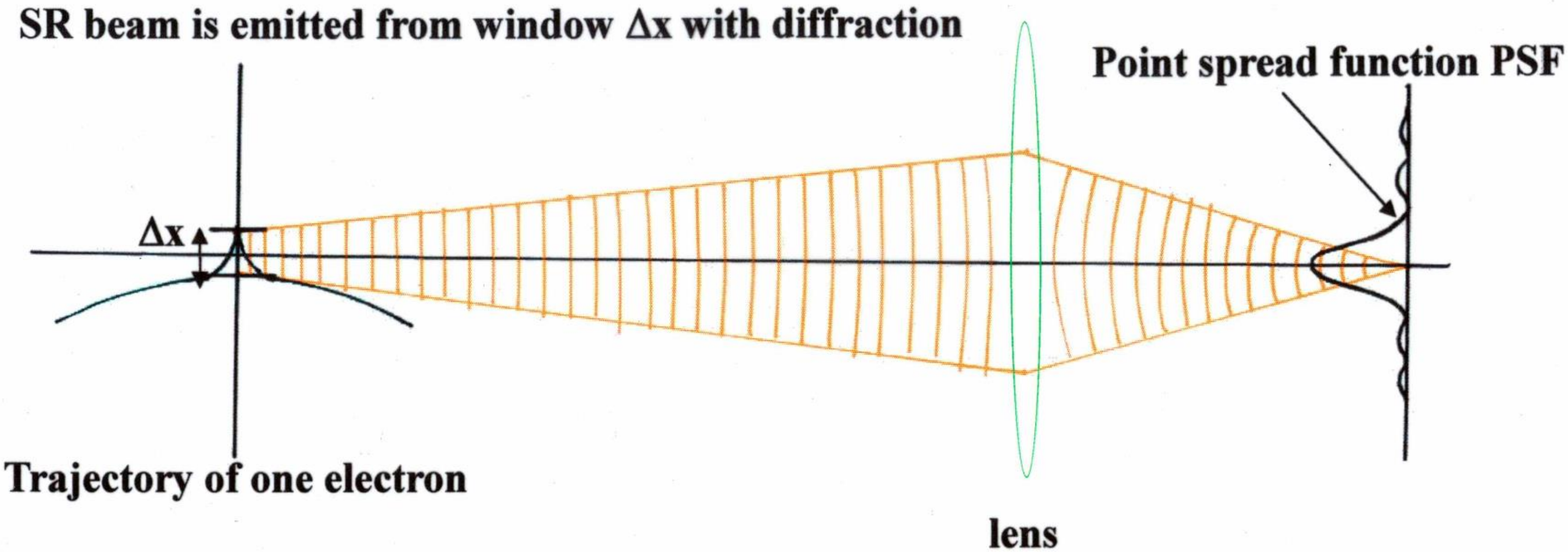


(b)

Simple understanding for single mode of photon in bending radiation (according to K.J.Kim's paradigm)

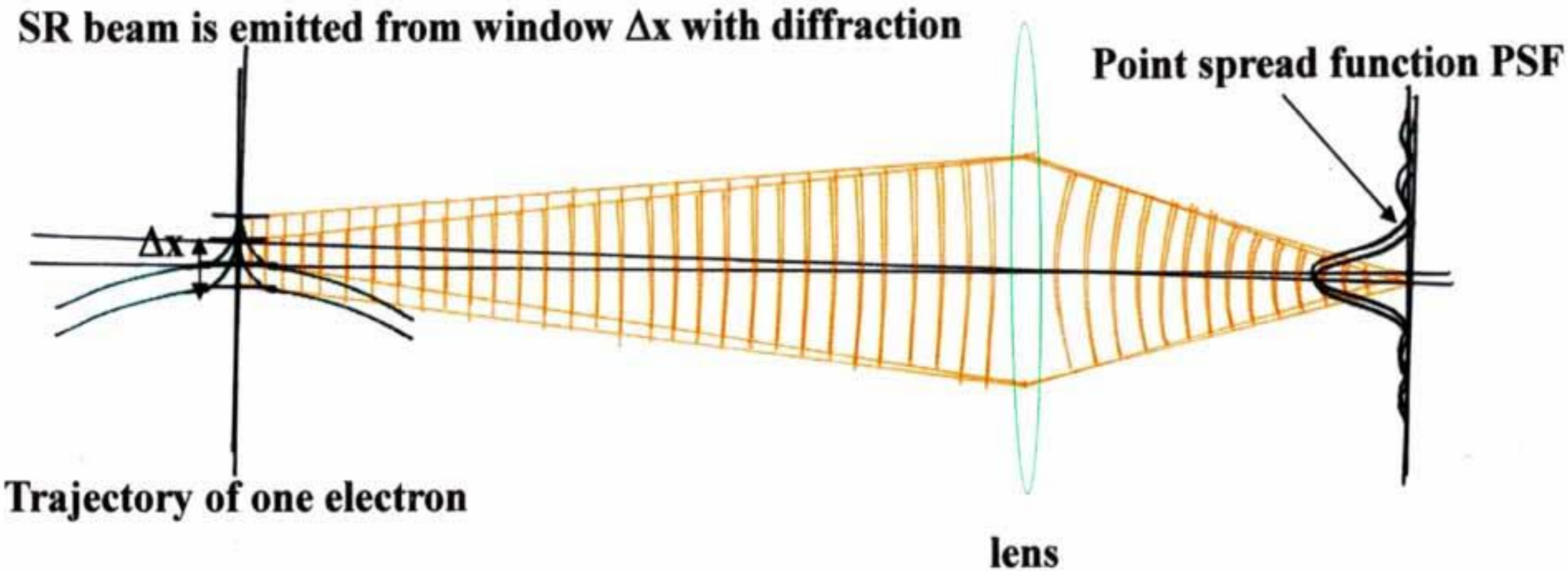


A convenient representation (two dimensional) of SR by K.J. Kim



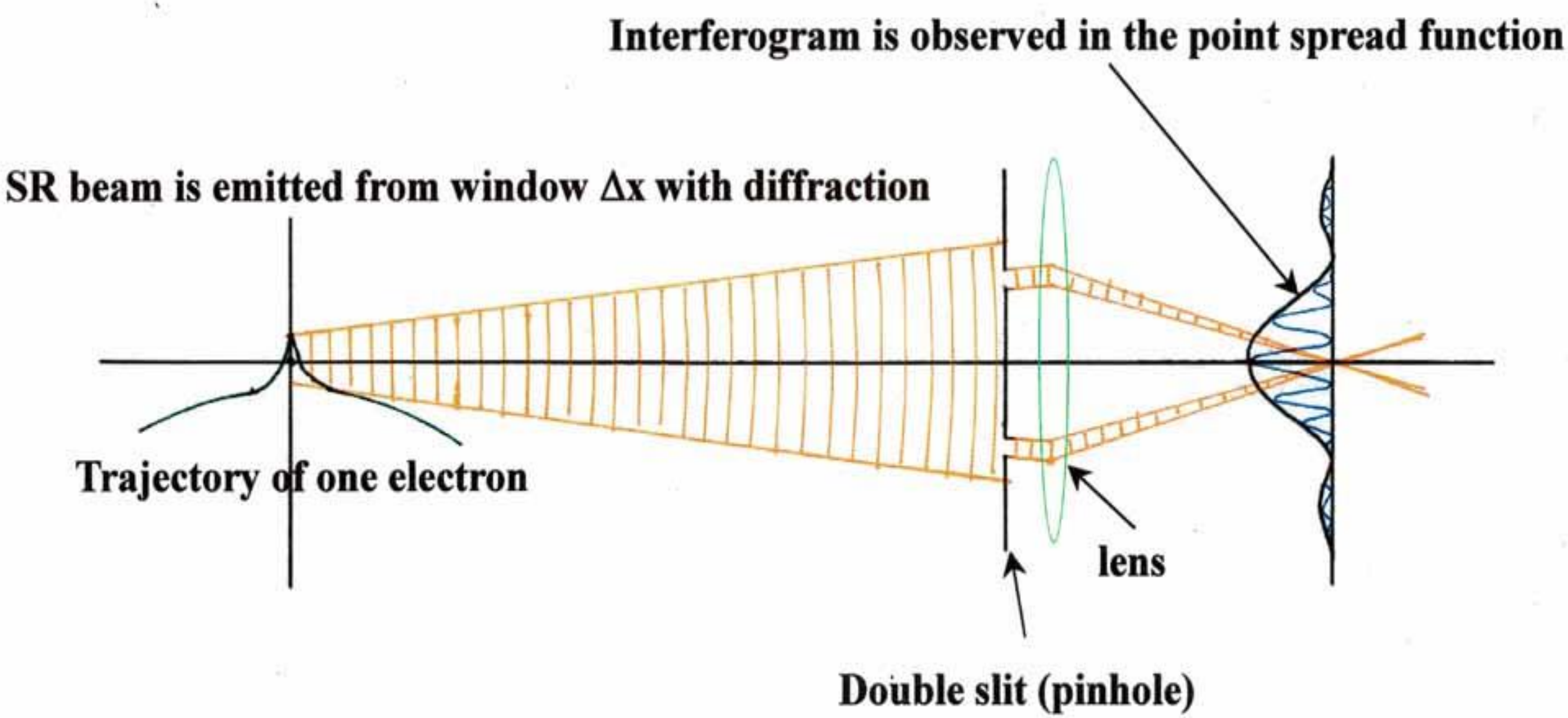
Significant temporal squeezing occurs in a small portion of the electron trajectory around a point tangent to the direction of observation. A single mode of one photon is radiated as a plane wave from the window Δx and it propagates with diffraction, and makes PSF at the focusing point.

A convenient representation (two dimensional) of SR by K.J. Kim



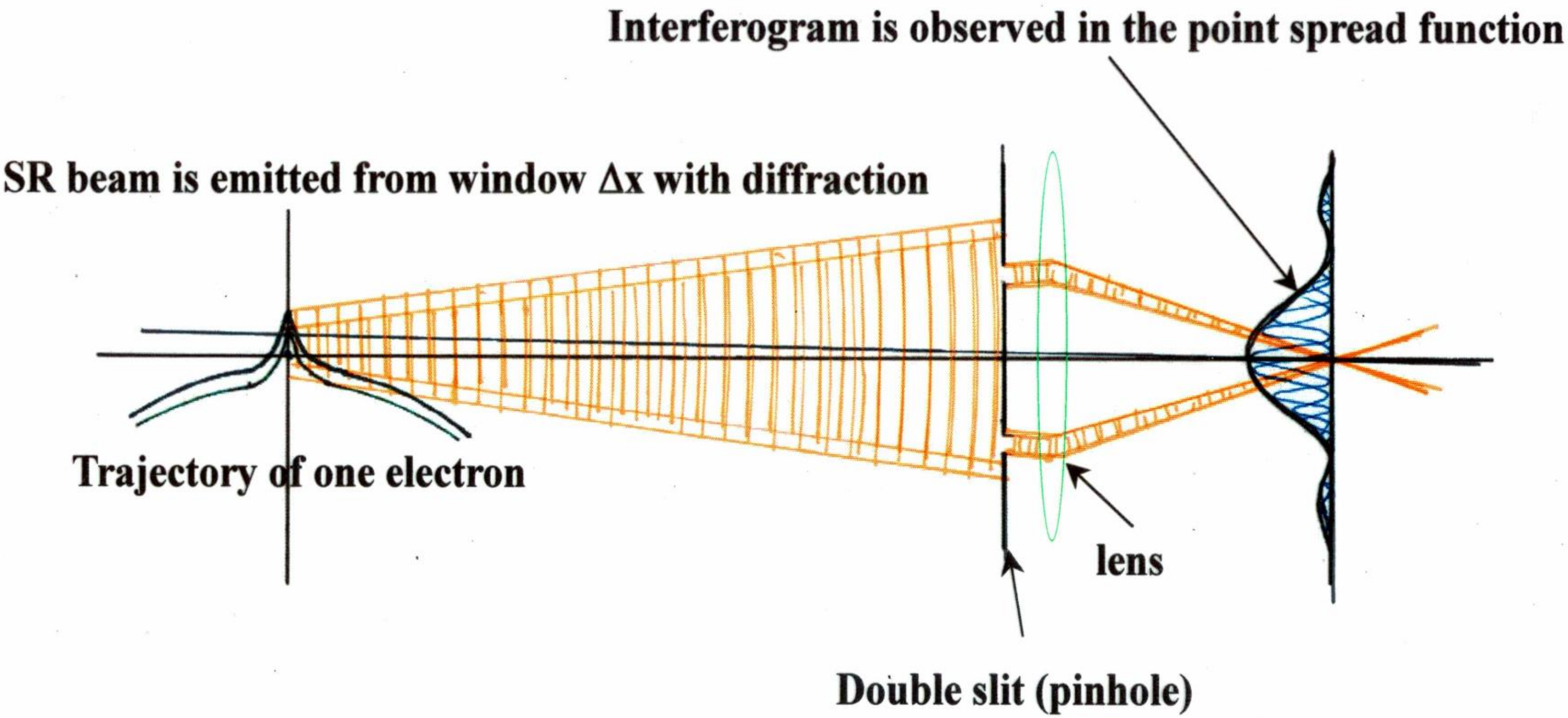
Significant temporal squeezing occurs in a small portion of the electron trajectory around a point tangent to the direction of observation. A single mode of one photon is radiated as a plane wave from the window Δx and it propagates with diffraction, and makes PSF at the focusing point.

Interferogram observed by a wavefront sampling type interferometer



Two mode of one photon is sampled by a double slit and those produce a interferogram with contrast 1 in the PSF of slit aperture.

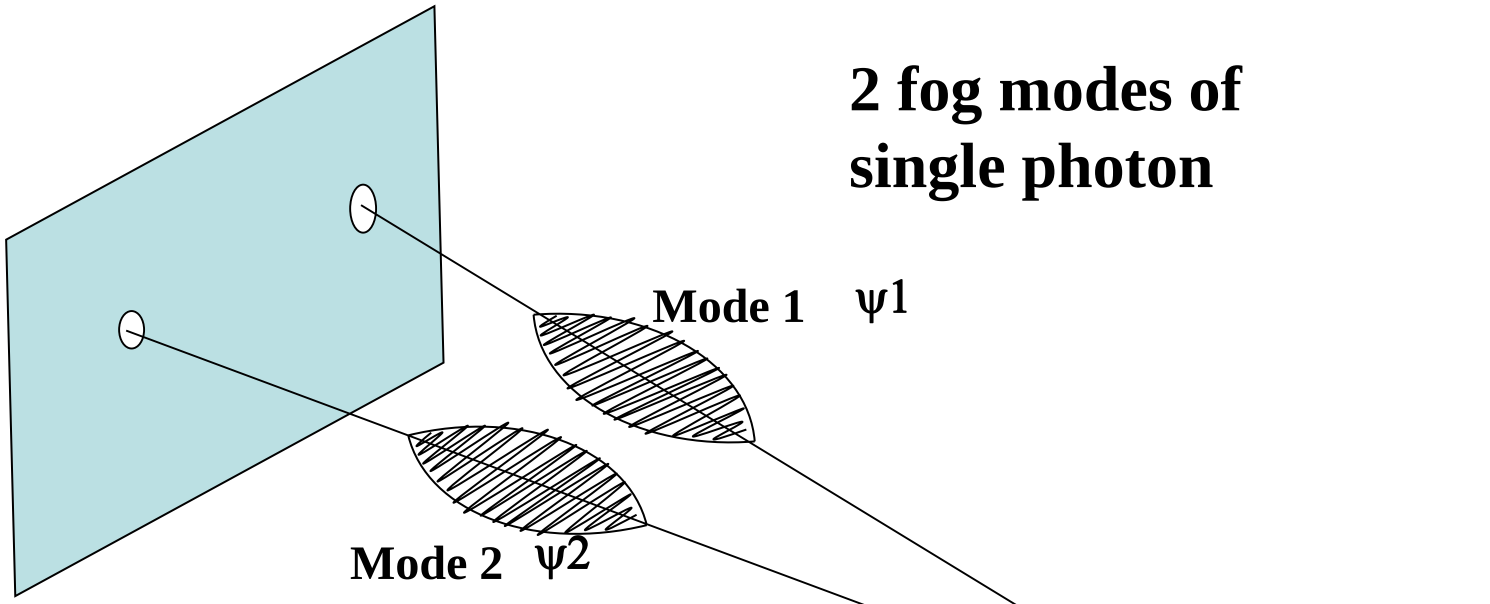
Interferogram observed by a wavefront sampling type interferometer



3-2. Theoretical resolution of interferometry

Uncertainty principle in phase of light

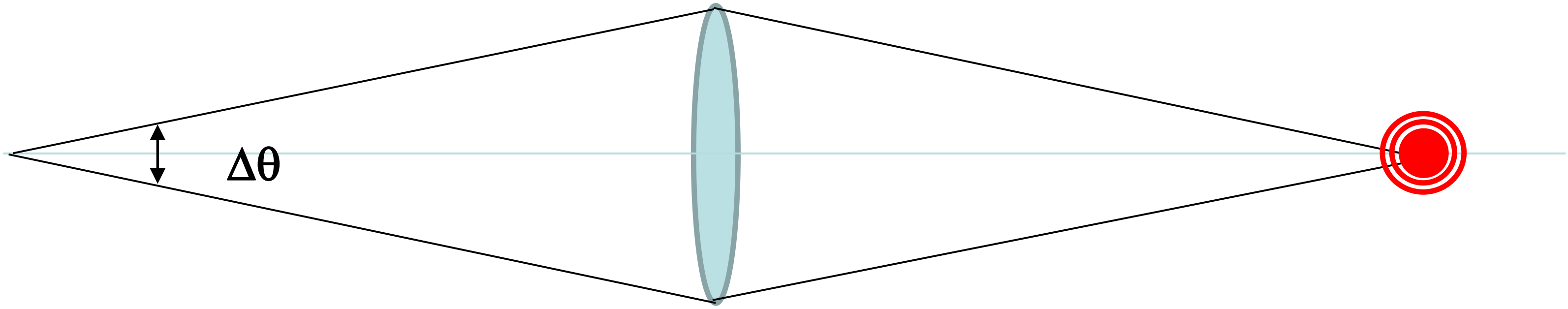
Function of the 1st order interferometer



2 fog modes of single photon

Measure the correlation of light phase in two modes

Uncertainty principal in imaging.



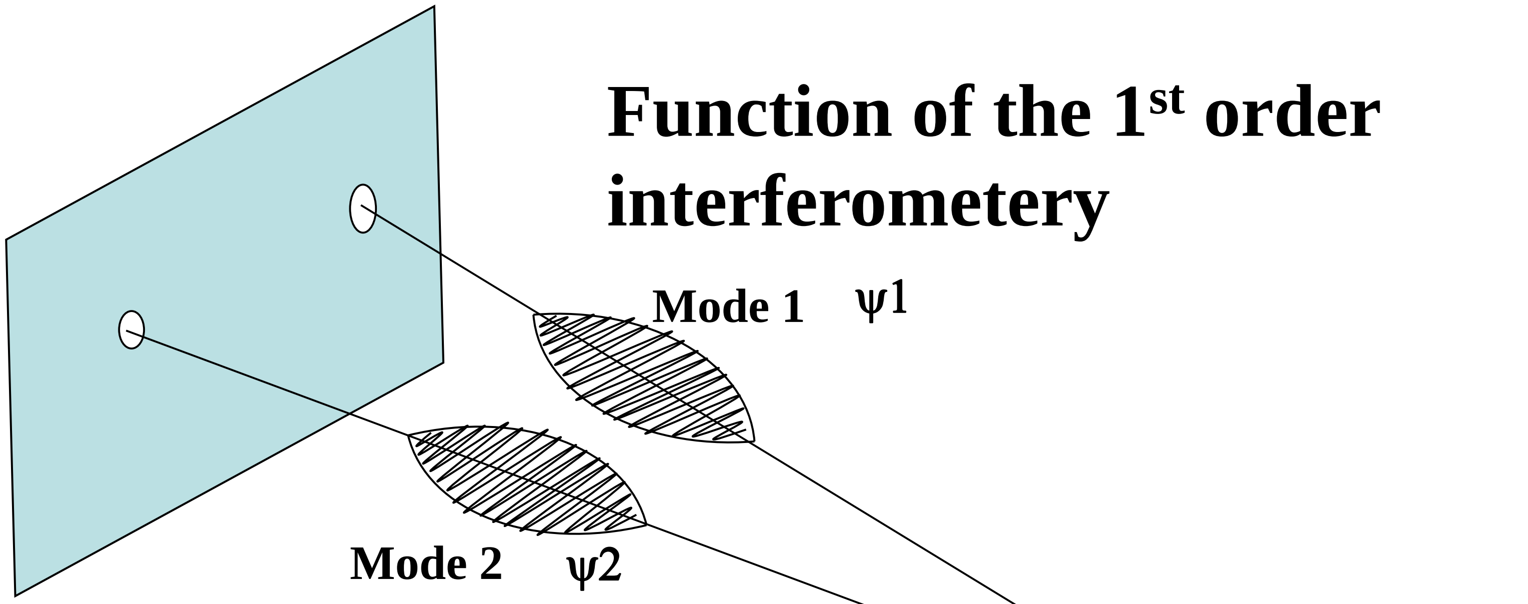
$\Delta\theta/\lambda \cdot \Delta x \geq 1$

So, large opening of light will necessary to obtain a good spatial resolution.

What is Uncertainty principal in interferometry ?

Uncertainty principal in interferometry

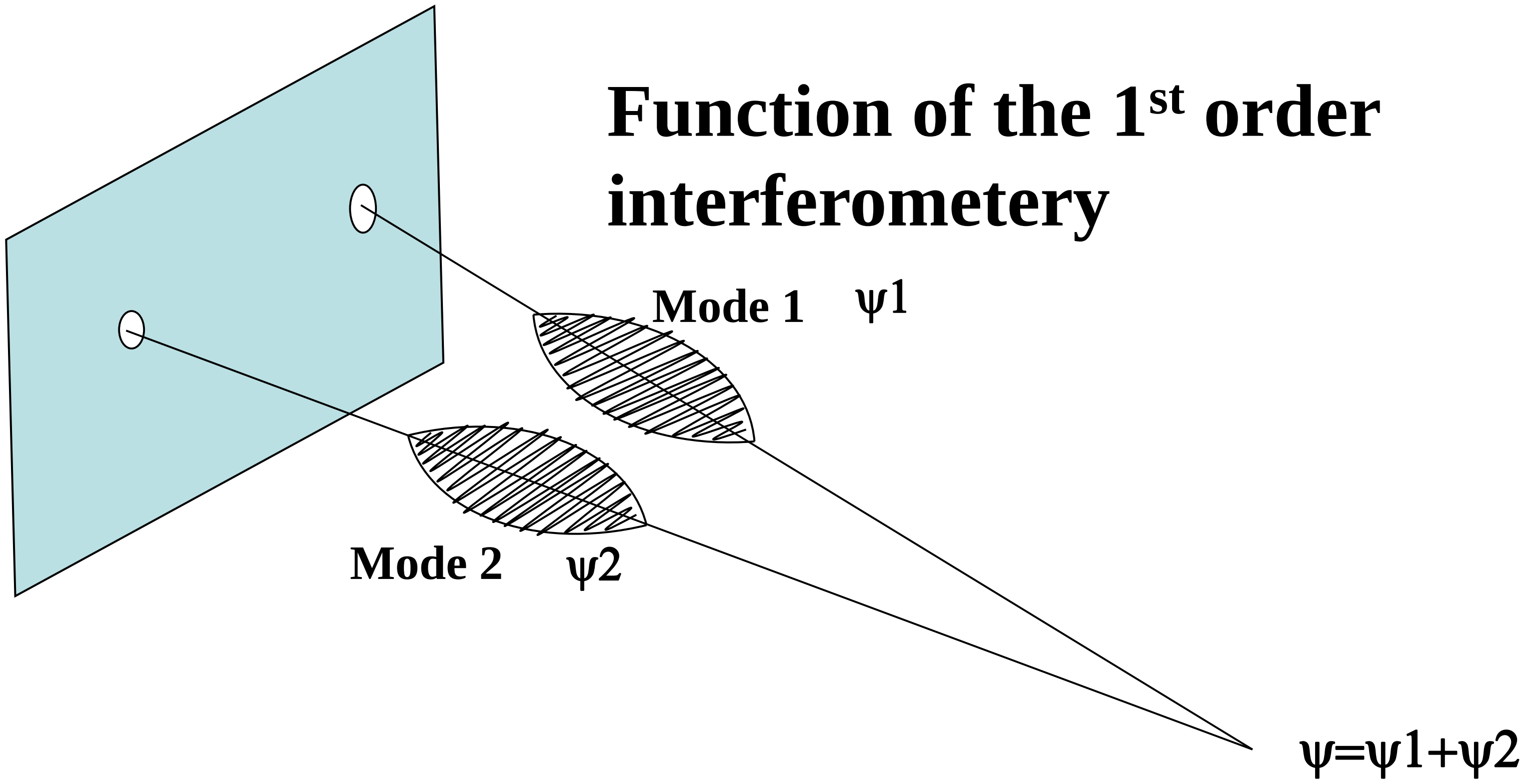
Function of the 1st order interferometery



Measure the correlation of light phase in two modes

Uncertainty principal in interferometry

Function of the 1st order interferometery



Measure the correlation of light phase in two modes ⇨ **Uncertainty in Phase $\Delta\phi$**

The interference fringe will be smeared by the uncertainty of phase.

$$I(y,D) = (I_1 + I_2)\left\{\sin c\left(\frac{\pi a y}{\lambda f}\right)\right\}^2 \left\{1 + \int_{\Delta\phi} \cos(kD\frac{y}{f} + \phi)\xi(\phi)d\phi\right\}$$

According to quantum optics,

Uncertaity priciple concerning to phase is given by

$$\Delta\phi \cdot \Delta N \geq 1/2$$

where ΔN is uncertainty of photon number.

Using the wavy aspect of photon in small number of photons, Forcibly ;

From uncertainty principal

$$\Delta\phi\cdot\Delta N\geq 1/2,$$

then,

$$\Delta\phi\geq 1/(2\cdot\Delta N).$$

Even in the case of coherent mode, interference fringe will be smeared by the uncertainty of phase.

$$I(y,D)=(I_1+I_2)\left\{\sin c\left(\frac{\pi a y}{\lambda f}\right)\right\}^2\left\{1+\int_{\Delta\phi}\cos(kD\frac{y}{f}+\phi)\xi(\phi)d\phi\right\}$$

Using the wavy aspect of photon in small number of photons, Forcibly ;

From uncertainty principal

$$\Delta\phi \cdot \Delta N \geq 1/2,$$

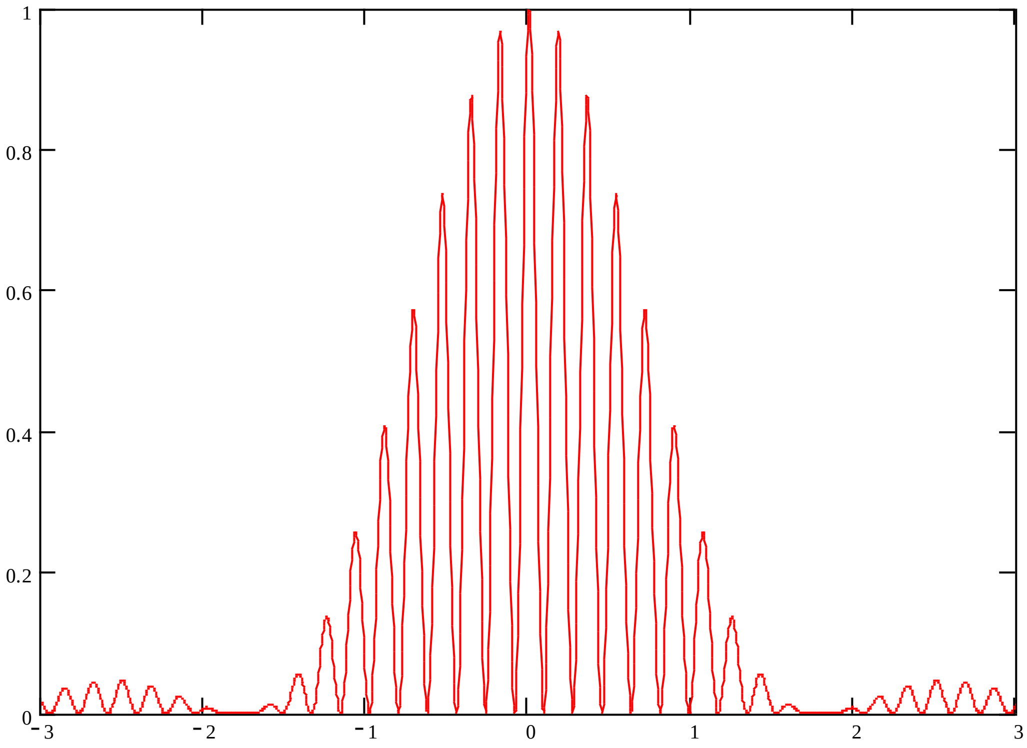
then,

$$\Delta\phi \geq 1/2 \cdot \Delta N.$$

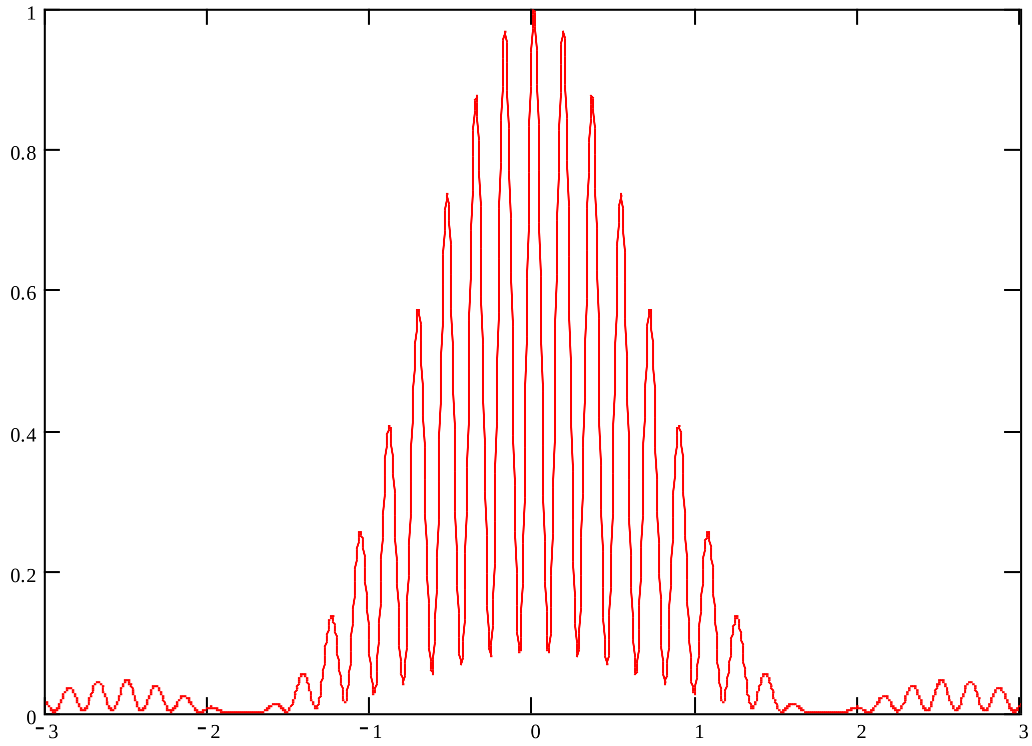
Even in the case of coherent mode, interference fringe will be smeared by the uncertainty of phase.

$$I(y, D) = \int_{\Delta\phi} (I_1 + I_2) \cdot \left\{ \sin c \left(\frac{\pi \cdot a \cdot y}{\lambda \cdot f} \right) \right\}^2 \left\{ 1 + \cos\left(k \cdot D \frac{y}{f} + \phi \right) \right\} d\phi$$

Interference fringe with no phase fluctuation

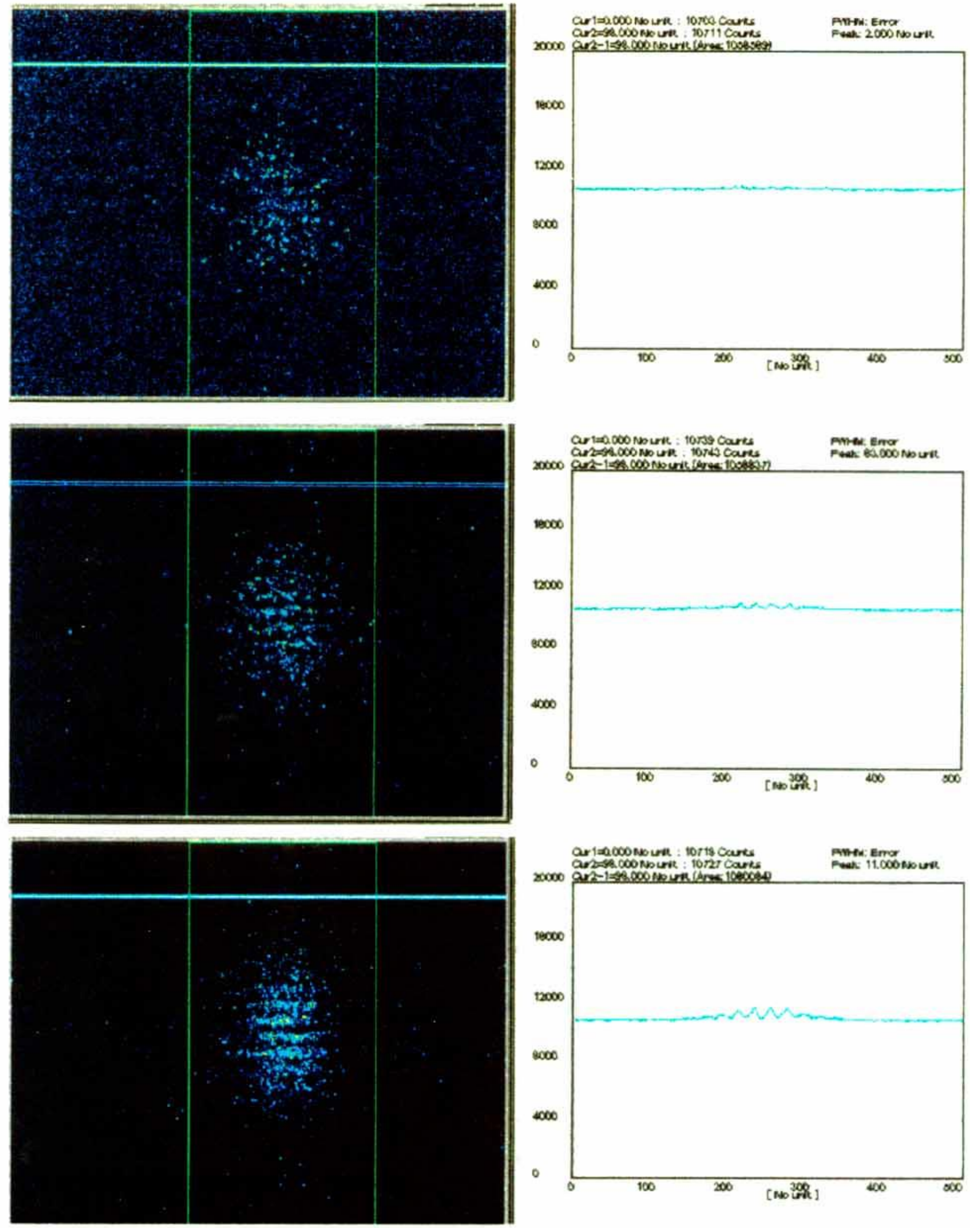


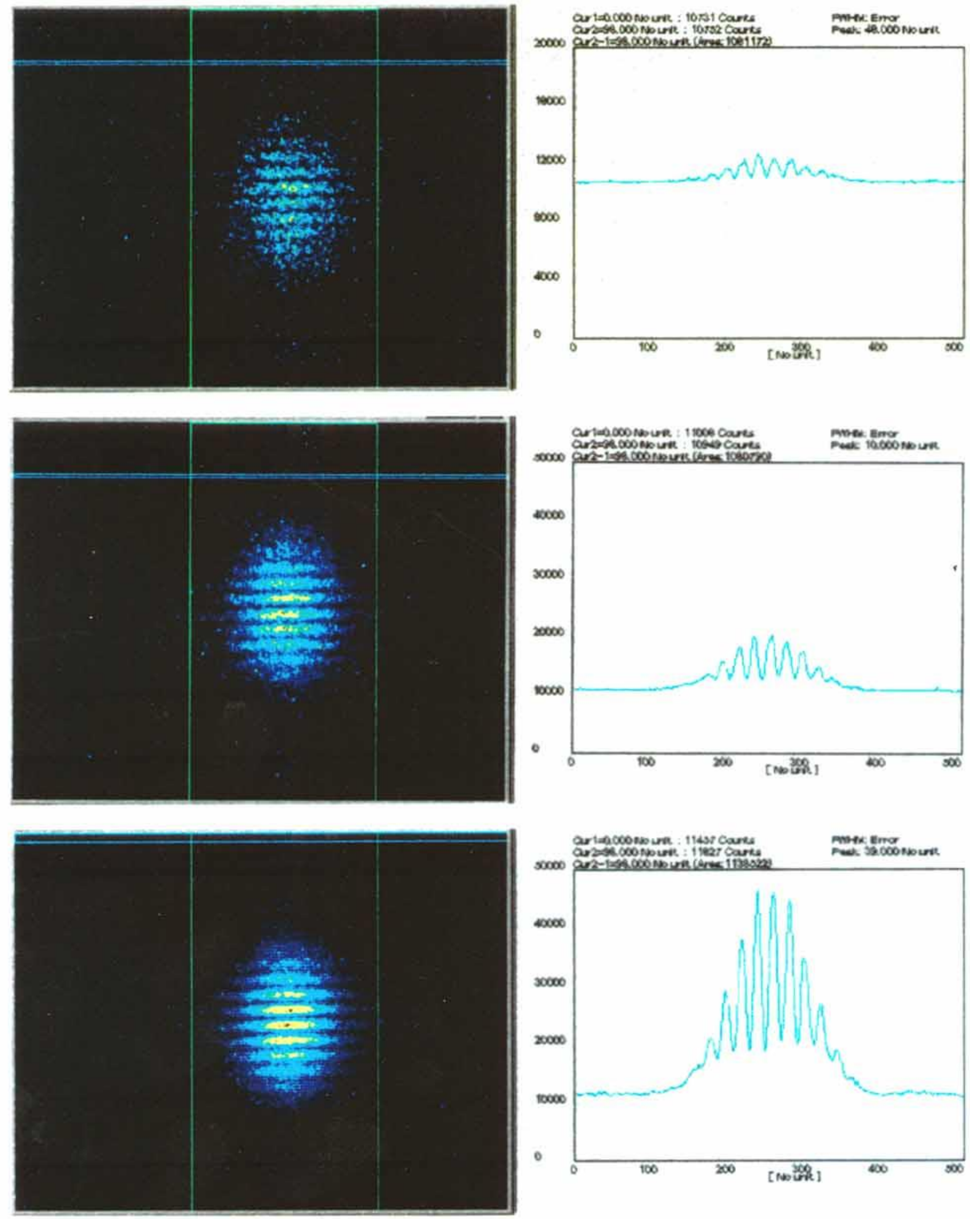
Interference fringe with uncertainty of phase π/2



We can feel the visibility of interference fringe will reduced by uncertainty of phase under the small number of photons. But actually, under the small number of photons, photons are more particle like, and difficult to see wave-phenomena.

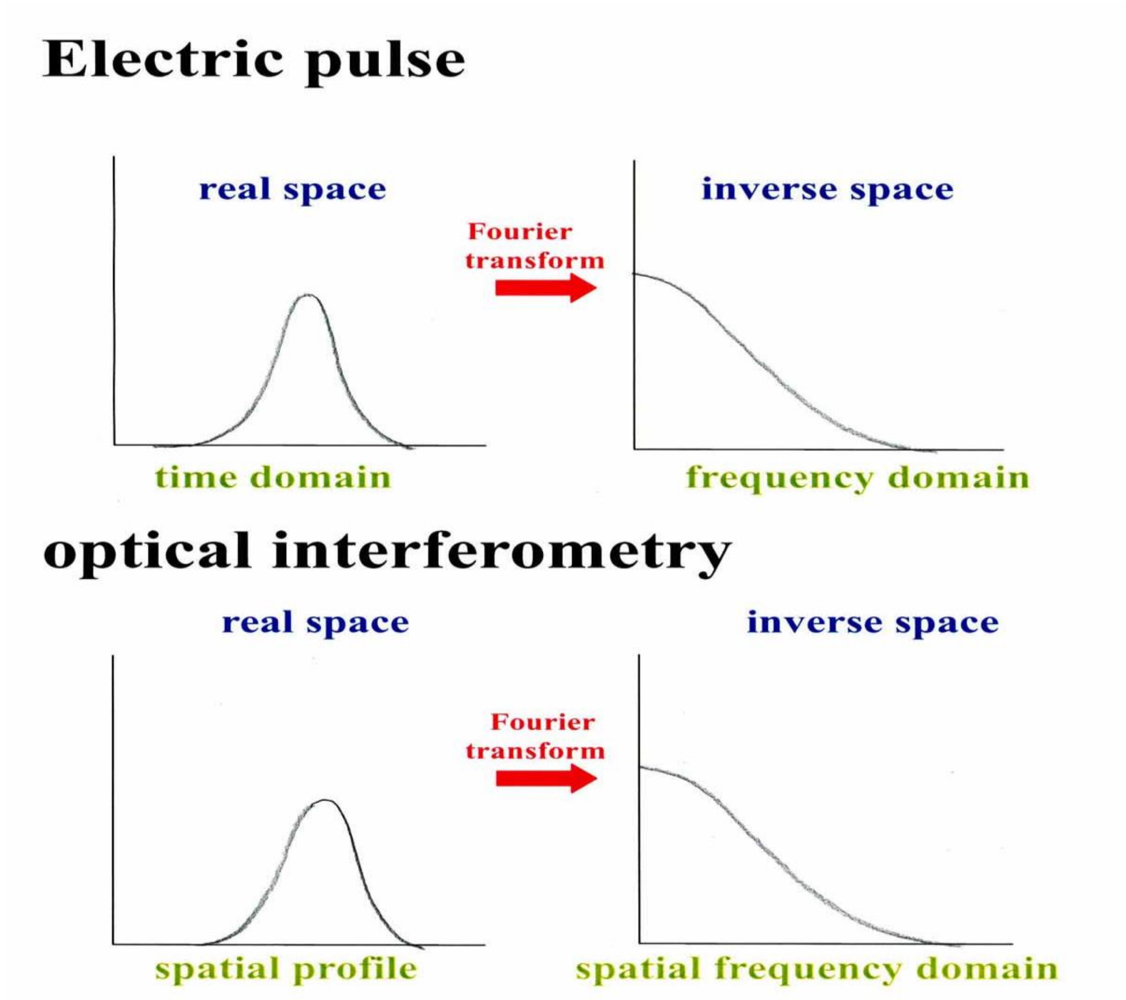
In actual case, we cannot observe interference fringe with small number of photons!





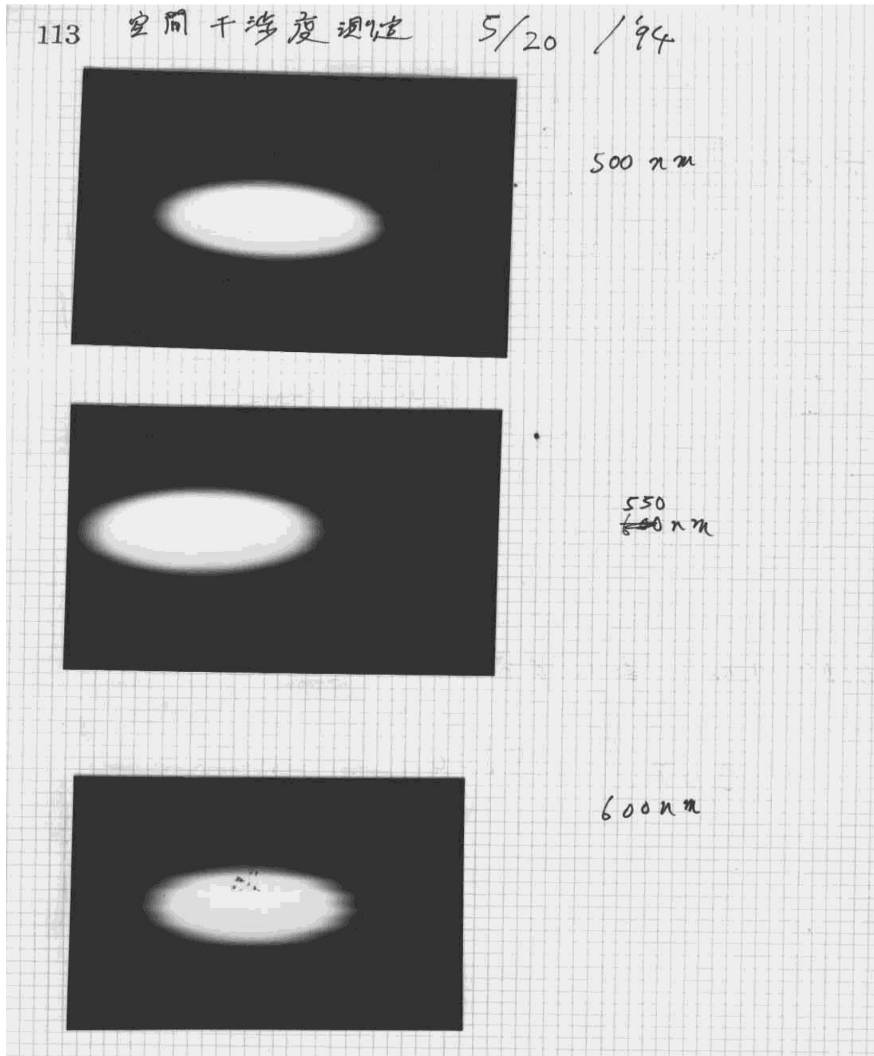
Cur1=0.000 No unit : 10731 Counts
Cur2=96.000 No unit : 10732 Counts
Cur2-1=96.000 No unit (Area: 1081172)
FWHM: Error
Peak: 48.000 No unit
20000
16000
12000
8000
4000
0
0
100
200
300
400
500
[No unit]
Cur1=0.000 No unit : 11006 Counts
Cur2=96.000 No unit : 10949 Counts
Cur2-1=96.000 No unit (Area: 1080790)
FWHM: Error
Peak: 10.000 No unit
50000
40000
30000
20000
10000
0
0
100
200
300
400
500
[No unit]
Cur1=0.000 No unit : 11437 Counts
Cur2=96.000 No unit : 11827 Counts
Cur2-1=96.000 No unit (Area: 1138522)
FWHM: Error
Peak: 39.000 No unit
50000
40000
30000
20000
10000
0
0
100
200
300
400
500
[No unit]

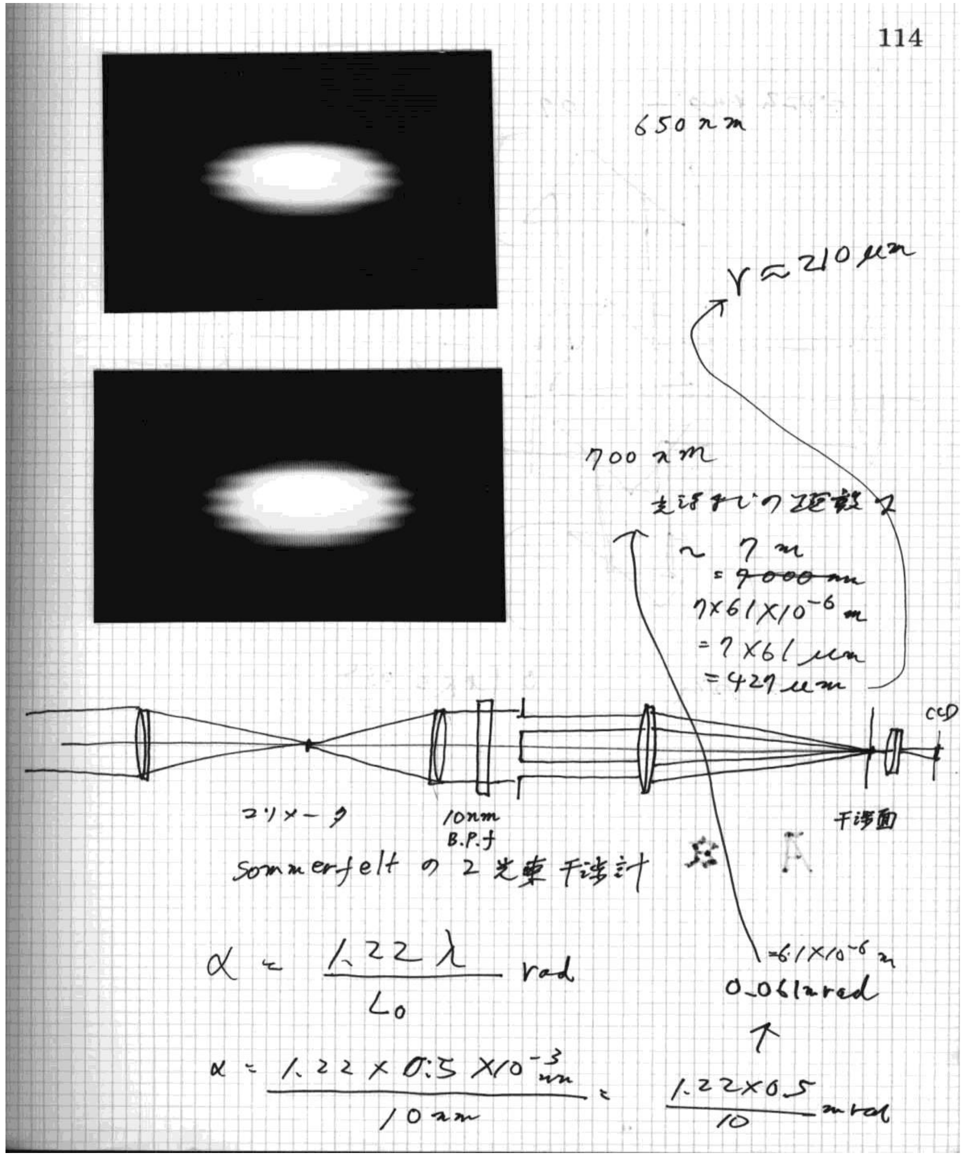
Comparison between electric circuit theory and optical interferometry



3-3. Early results at the Photon Factory

First experiment of the interferometry in 1994





Result of beam size is 210μm

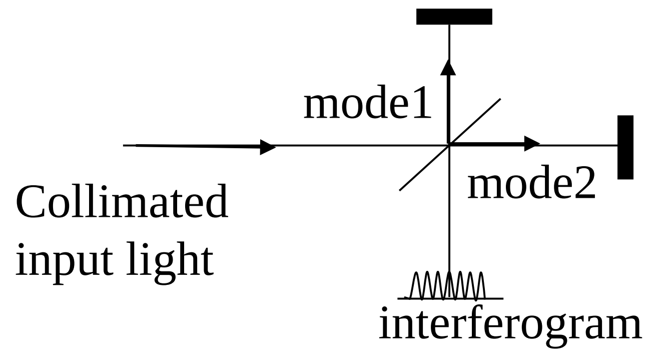
3-4. Spatial coherence measurement and its application for beam profile measurement

Choice of interferometer type

1. Intensity division type

Michelson type

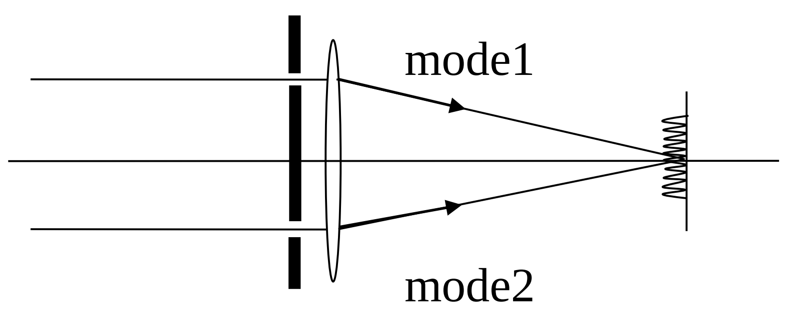
This type is suitable to measure the temporal coherence



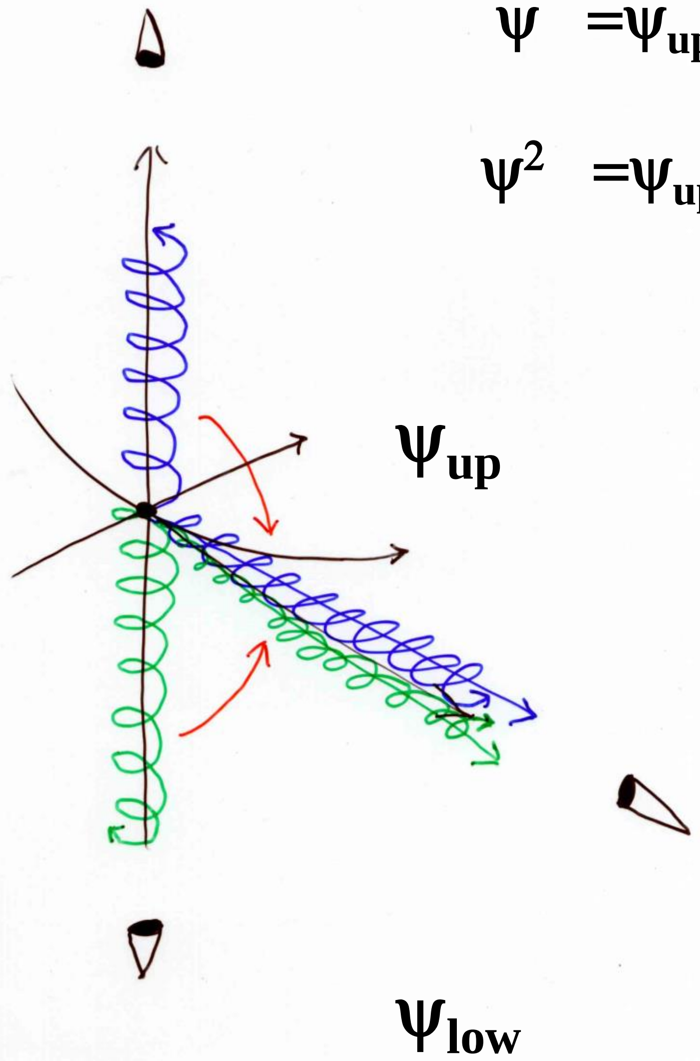
2. Wavefront division type

Rayleigh type

This type is suitable to measure the spatial coherence



Difference in horizontal and vertical polarized components.

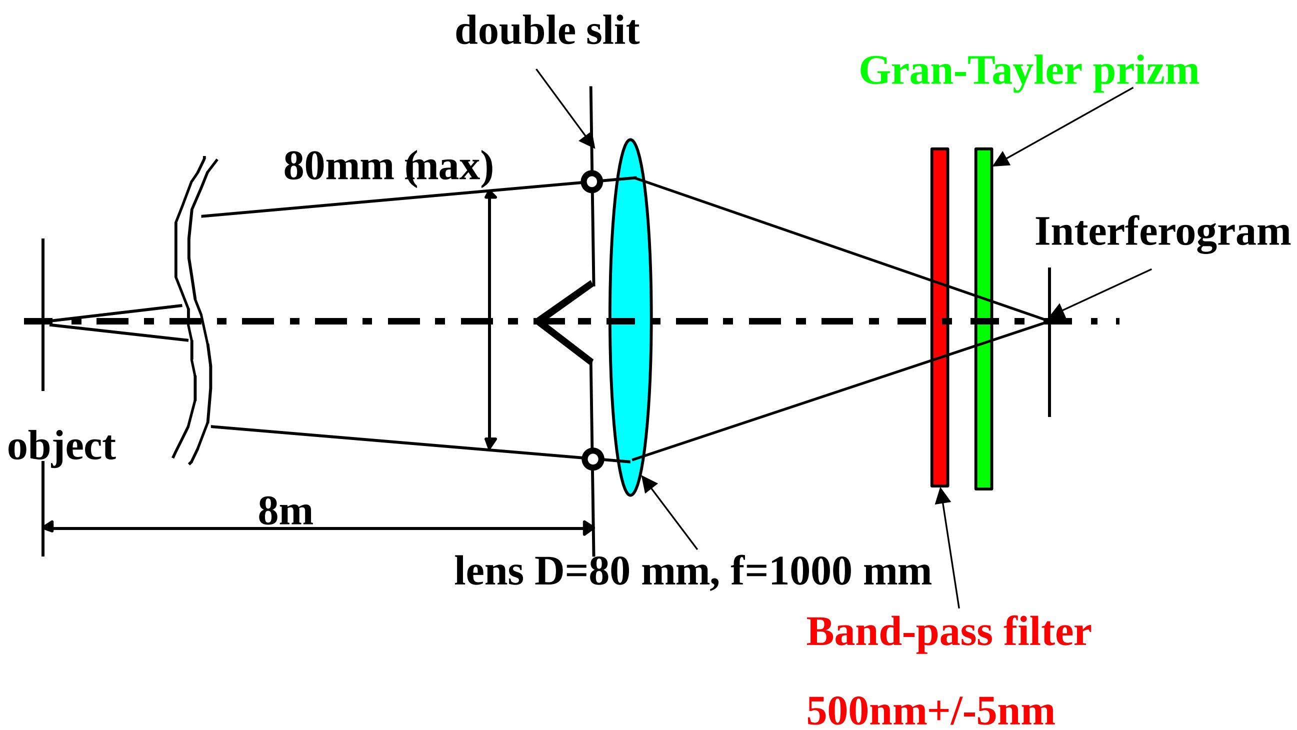


$\psi \; = \psi_{up} + \psi_{low}$ **one photon two modes**

$\psi^2 \; = \psi_{up}^2 + \psi_{low}^2$ **two independent modes**

Which is correct?

Design of interferometer. Dimensions are those used at the Photon Factory



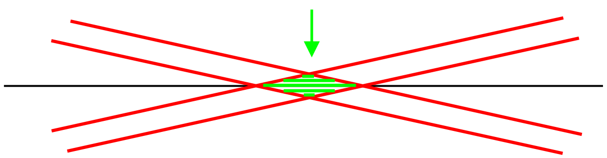
Important function of the objective lens ;

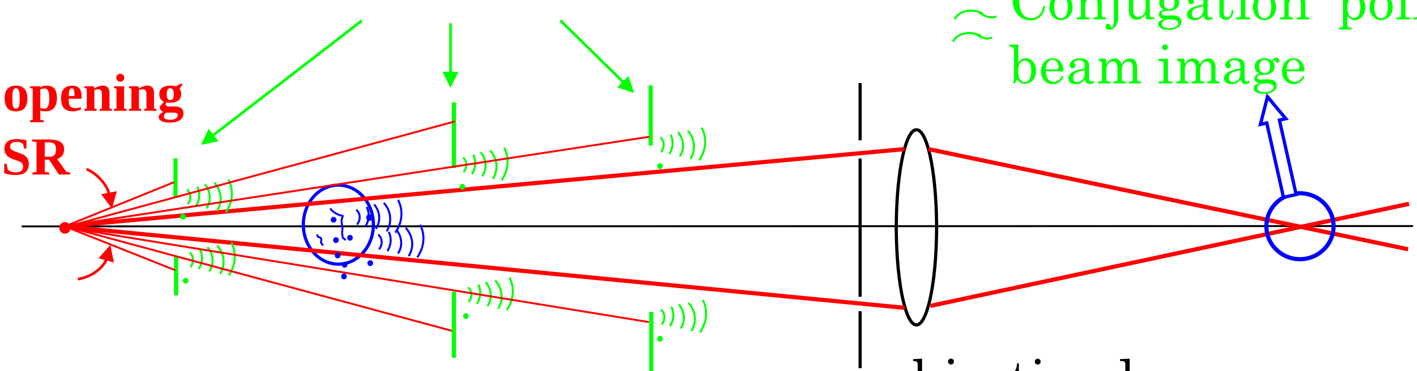
1. Localize the interfering volume to make the interferometer bright.

2. Elimination of parasitic modes coming into the interfering volume.

Depth of interfering volume is given by

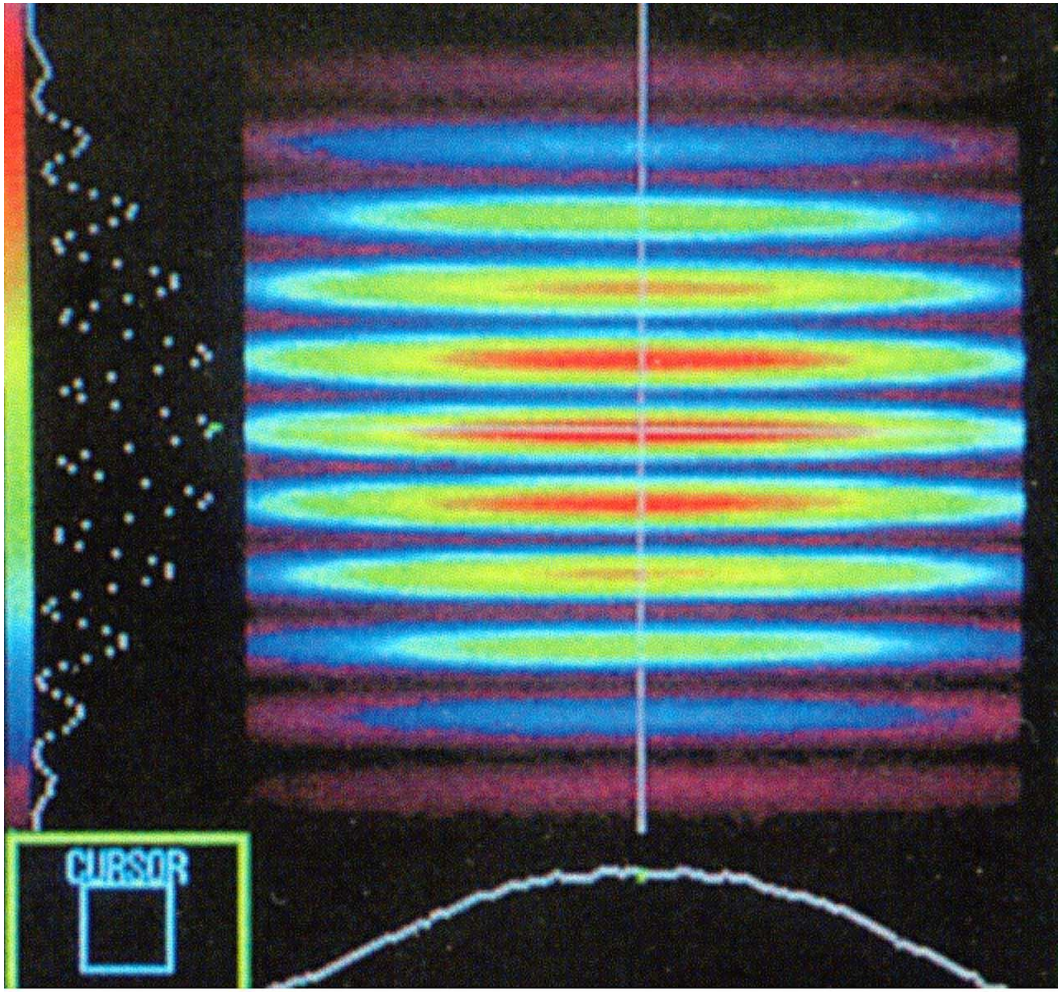
$$\Delta z \cong \frac{3.2\,\lambda}{\pi}\left(\frac{f}{a}\right)^2.$$



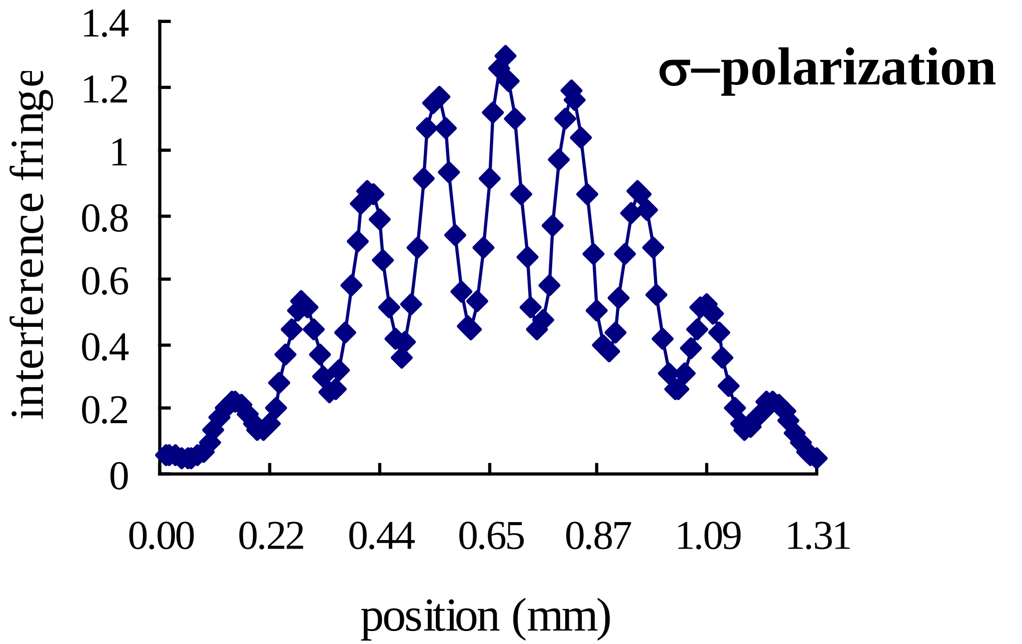


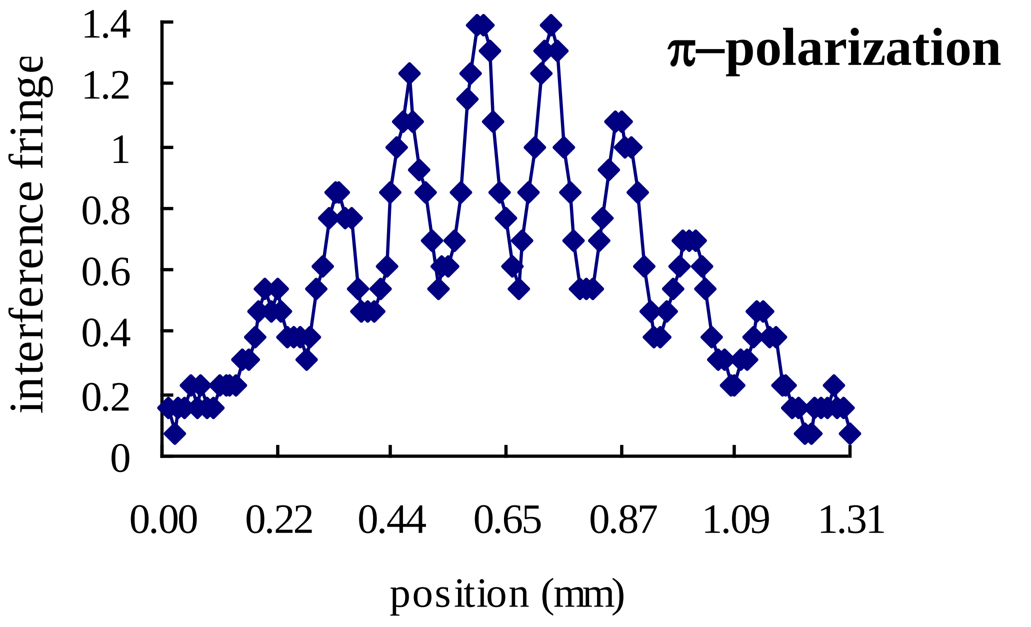
parasitic modes < 0.1% into the interfering volume

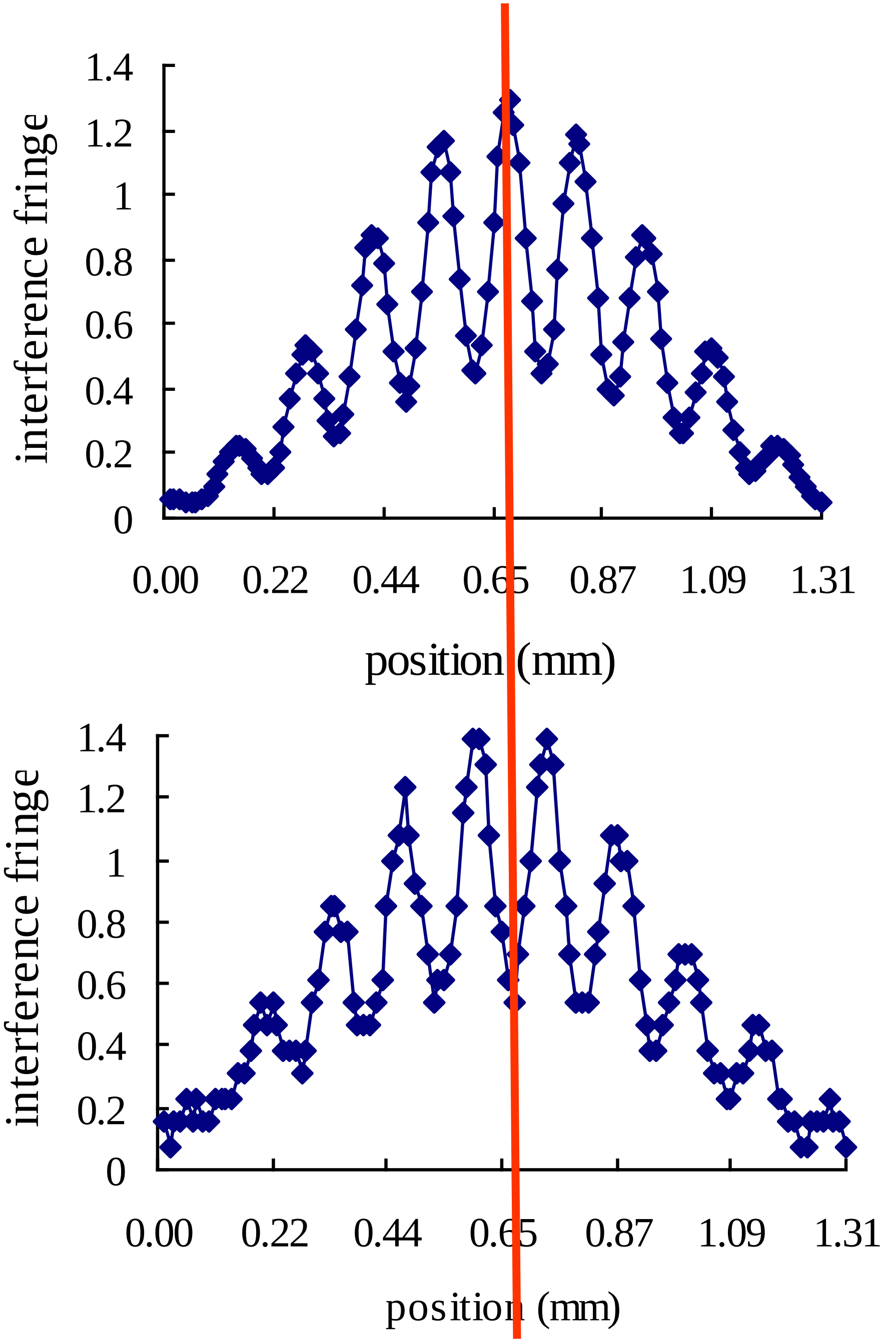
Typical interferogram in vertical direction at the Photon Factory. D=10mm



Comparison between experimental results

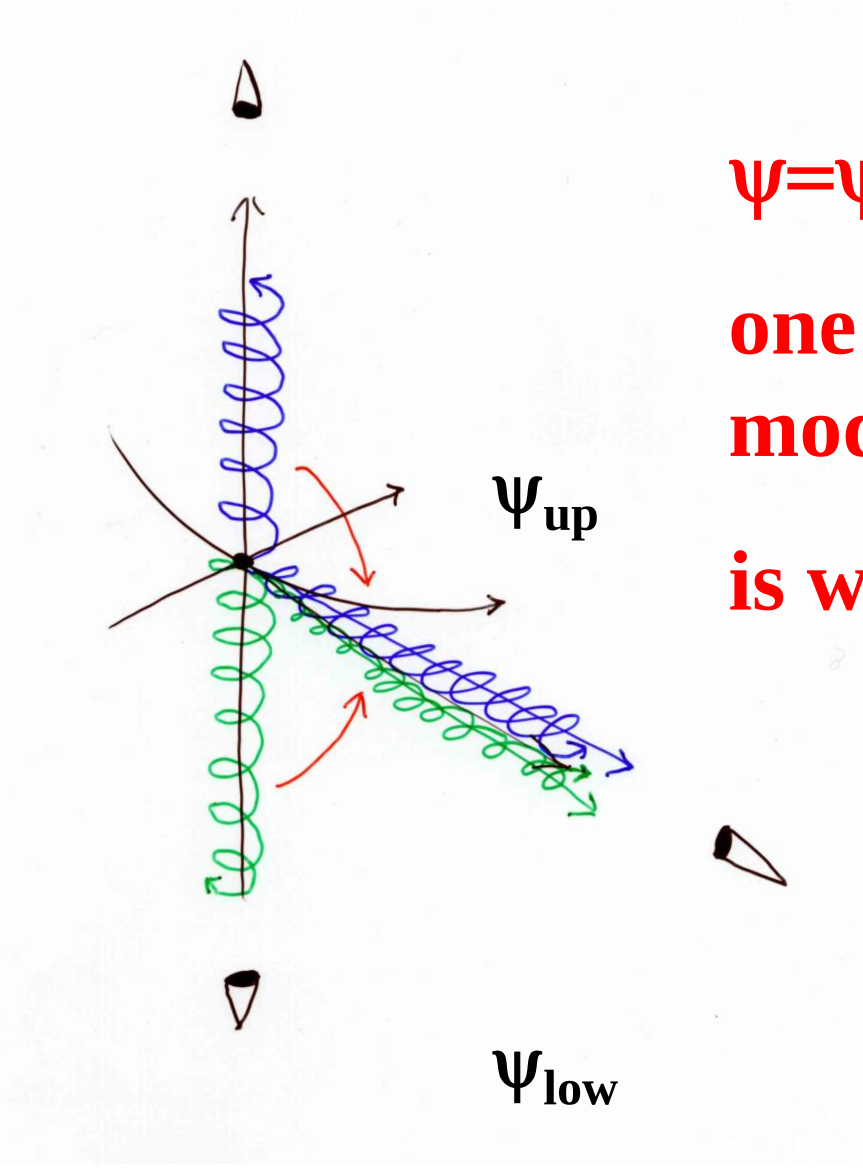






$\psi = \psi_{up} + \psi_{low}$

one photon two modes



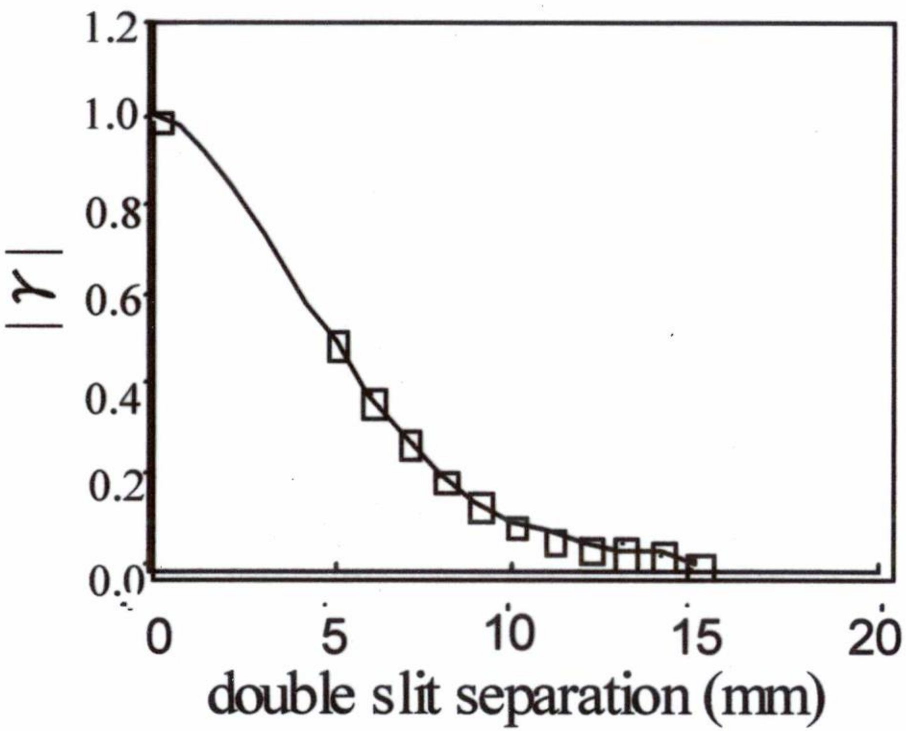
$\Psi = \Psi_{up} + \Psi_{low}$

one photon two modes

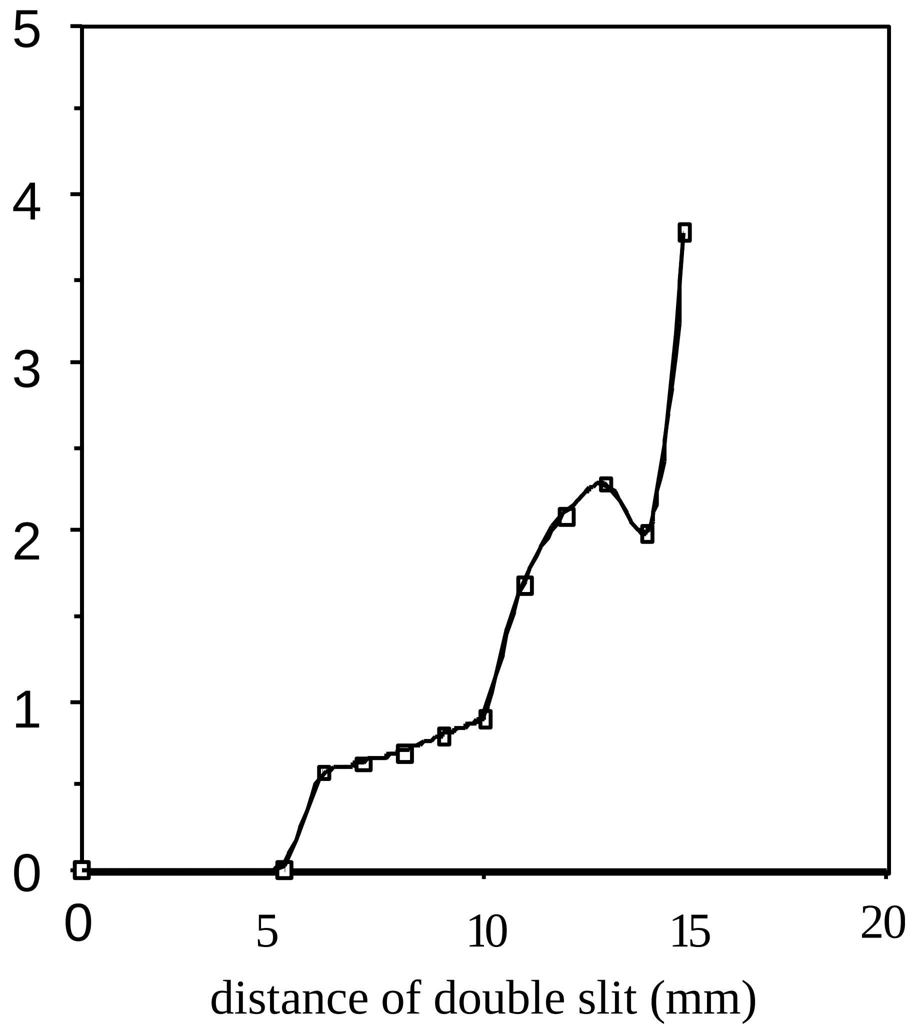
is winner

6-2. Result of spatial coherence measurement

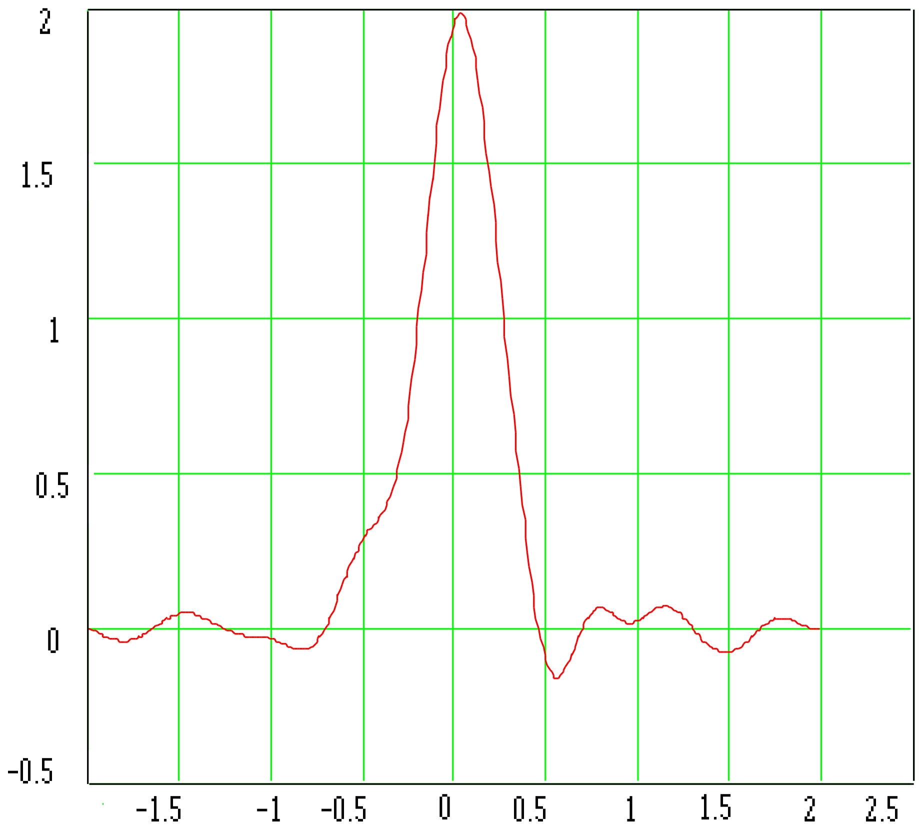
Absolute value of the complex degree of spatial coherence me asured at the Photon Factory KEK.



Phase of the complex degree of spatial coherence
vertical axis is phase in radian

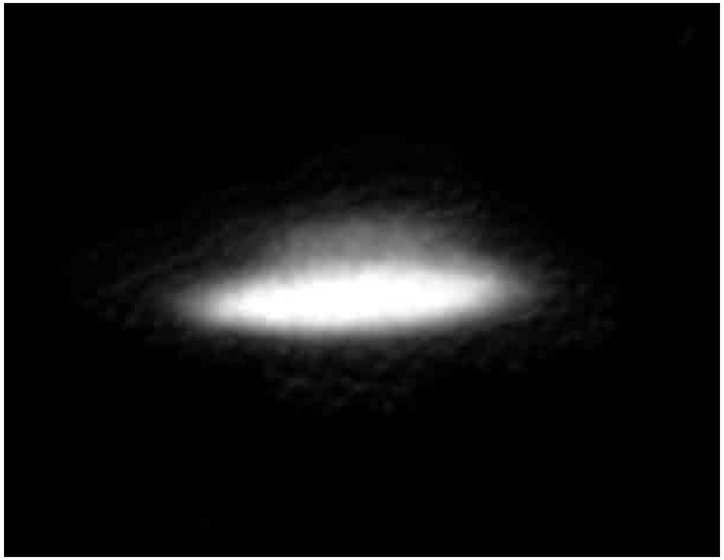


6-3. reconstruction of beam profile by Fourier transform

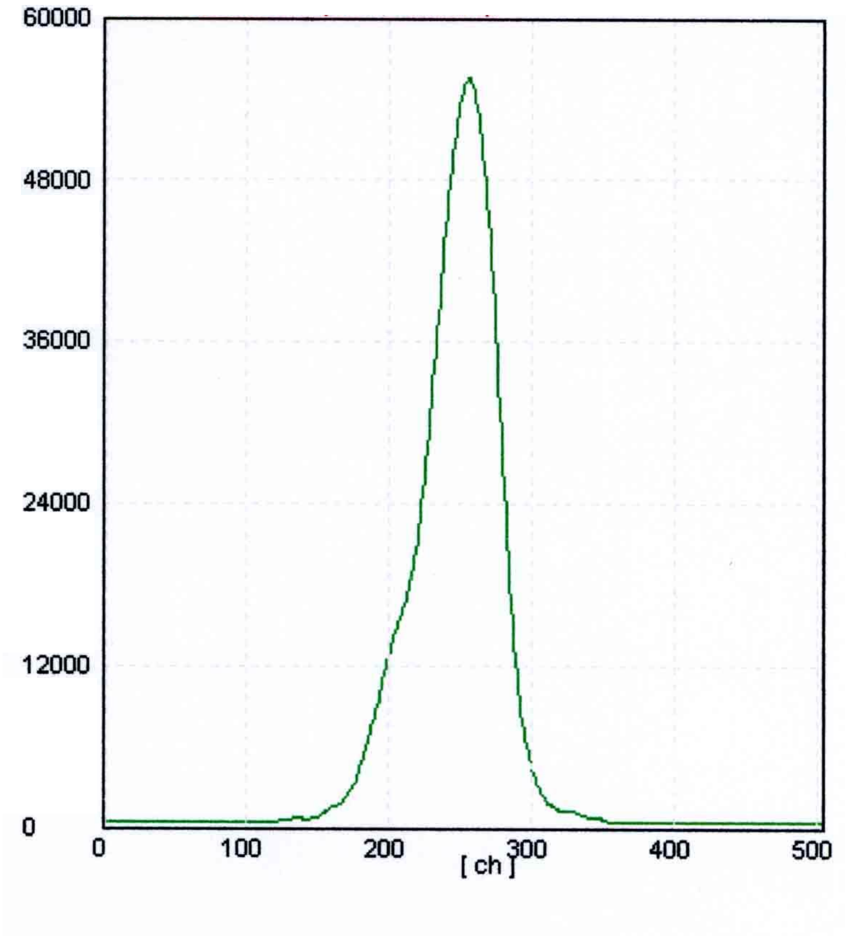


Vertical beam profile obtained by a Fourier transform of the complex degree of coherence.

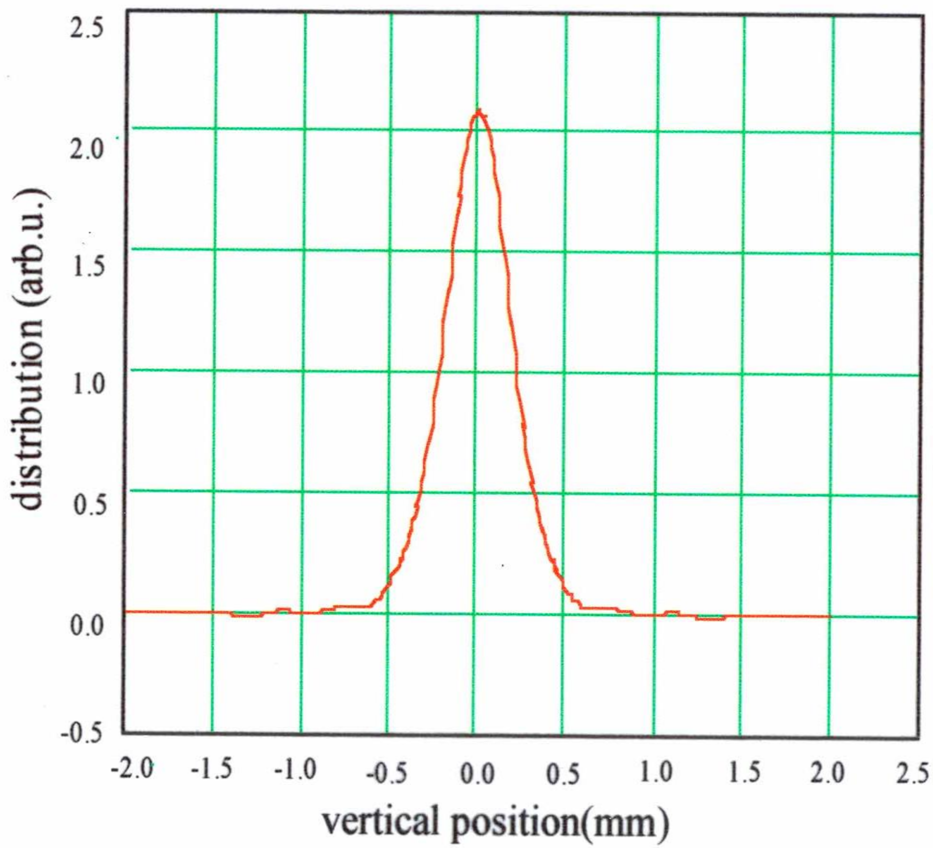
Comparison between image



Beam profile taken with an imaging system

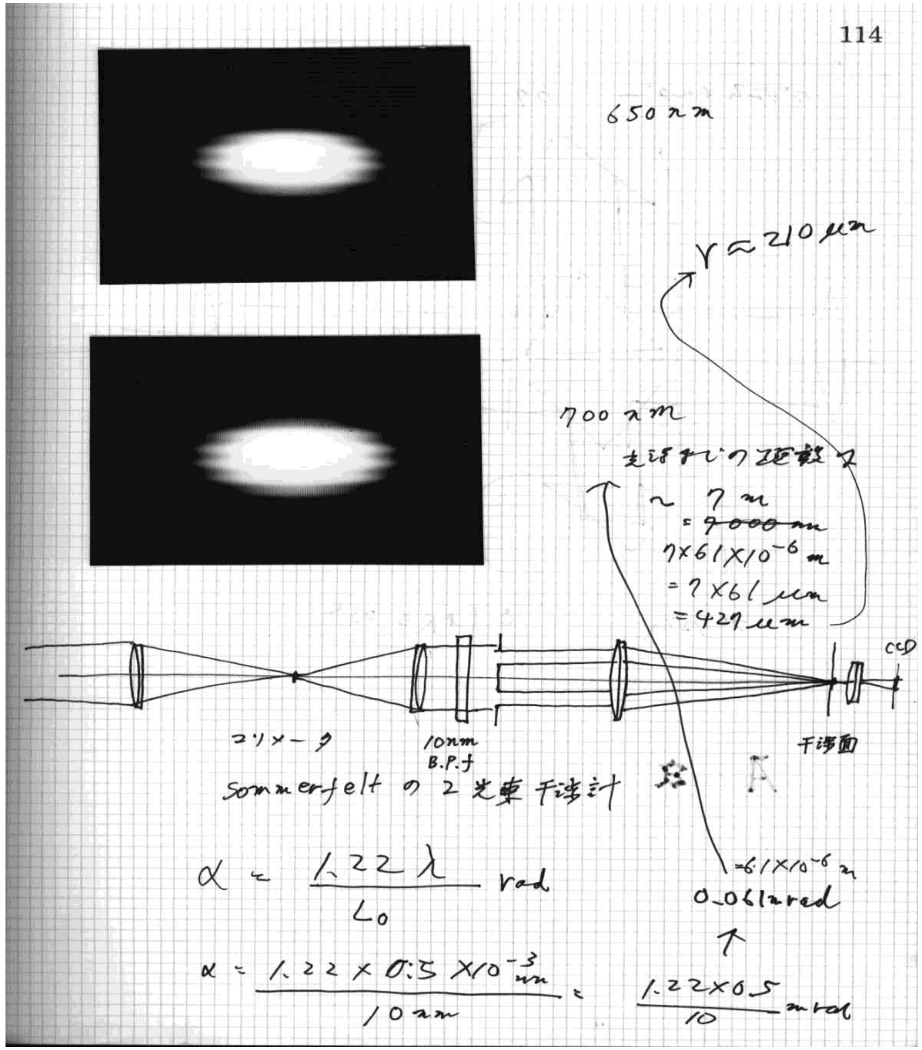


Vertical beam profile obtained by Fourier Cosine transform



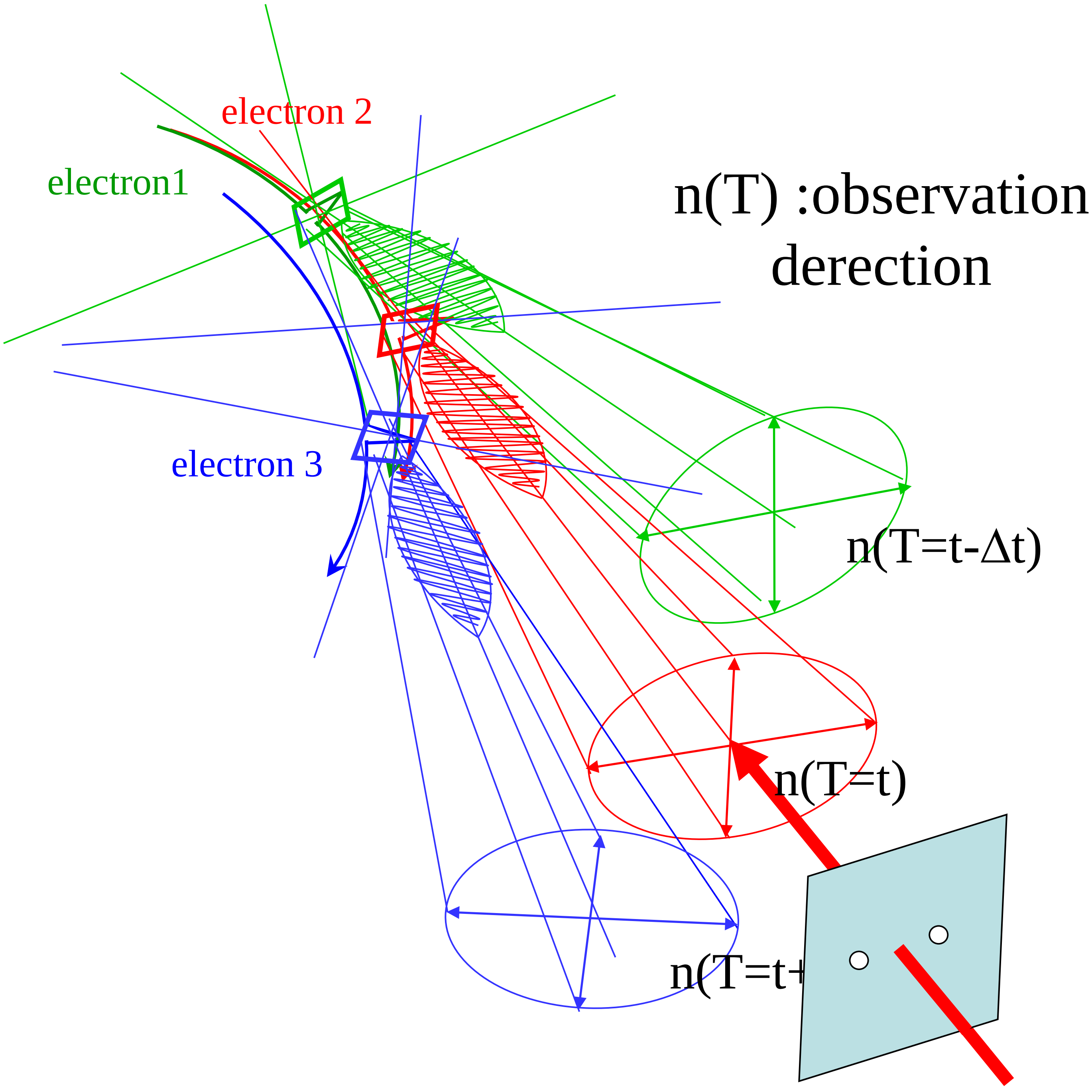
Result of beam size is 214μm

Result by imaging is 228μm



Result of beam size is 210μm

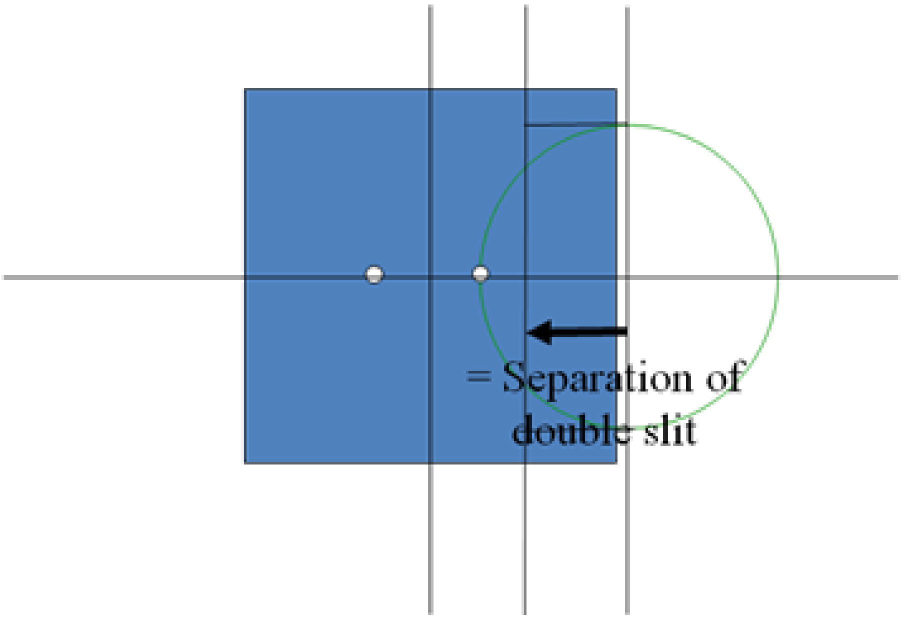
3-5. Effect of incoherent field depth into horizontal interferogram measurement

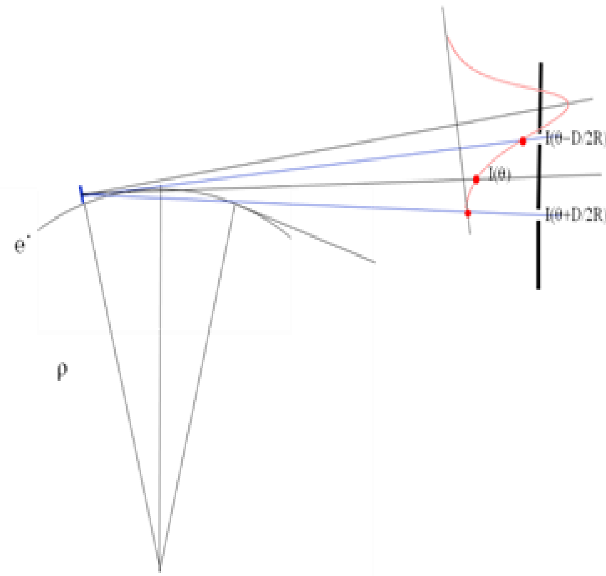


electron 2
electron1
electron 3
n(T) :observation derection
n(T=t-Δt)
n(T=t)
n(T=t+

Situation 1　One opening of double slit covers with SR

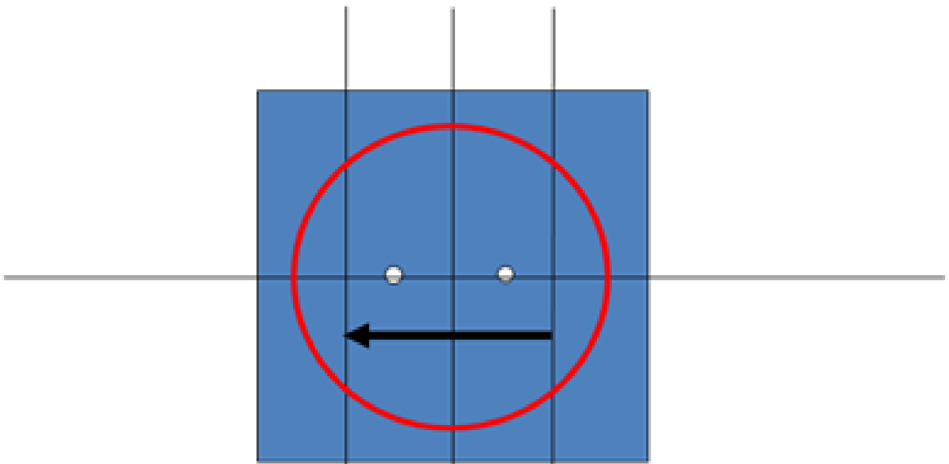
No interferogram

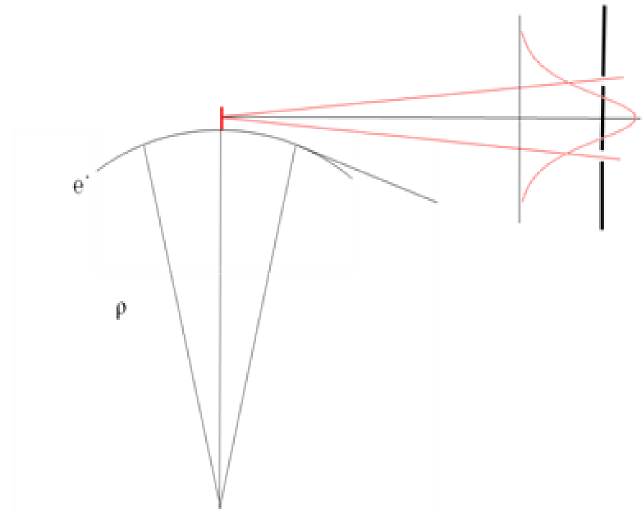




Situation 2 Two openings of double slit cover with wavepocket of SR

Interferogram will appear

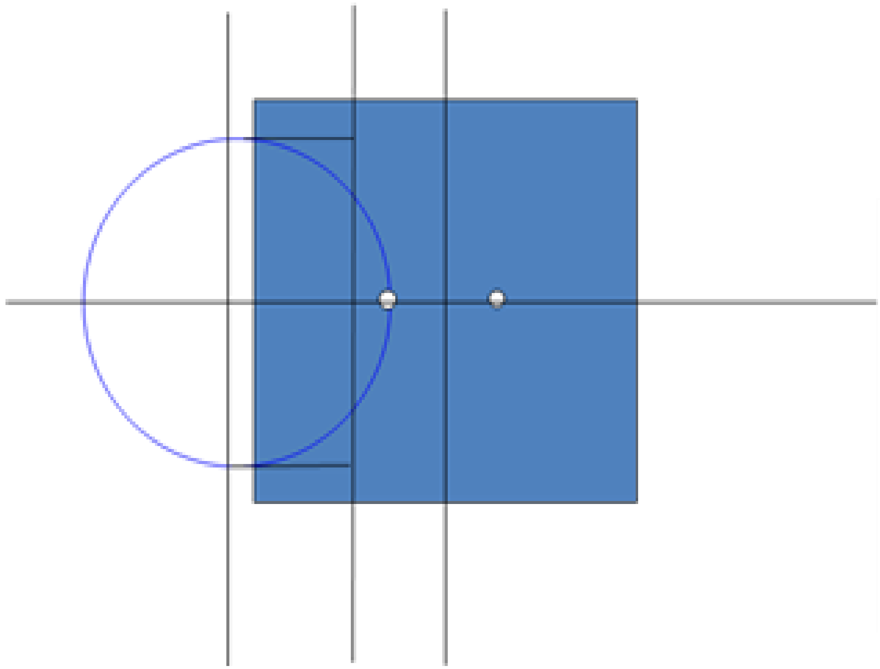


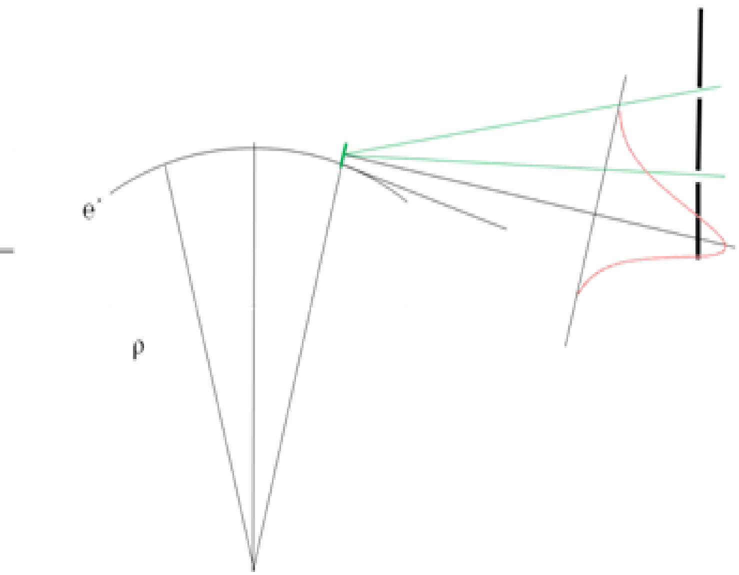


Intensity of interferogram is proportional to its average intensity at the openenings and It's contrast is reduced by intensity imbalance factor.

Situation 3 One opening of double slit again covers with SR

No interferogram





When the incoherent field depth <diffraction field depth

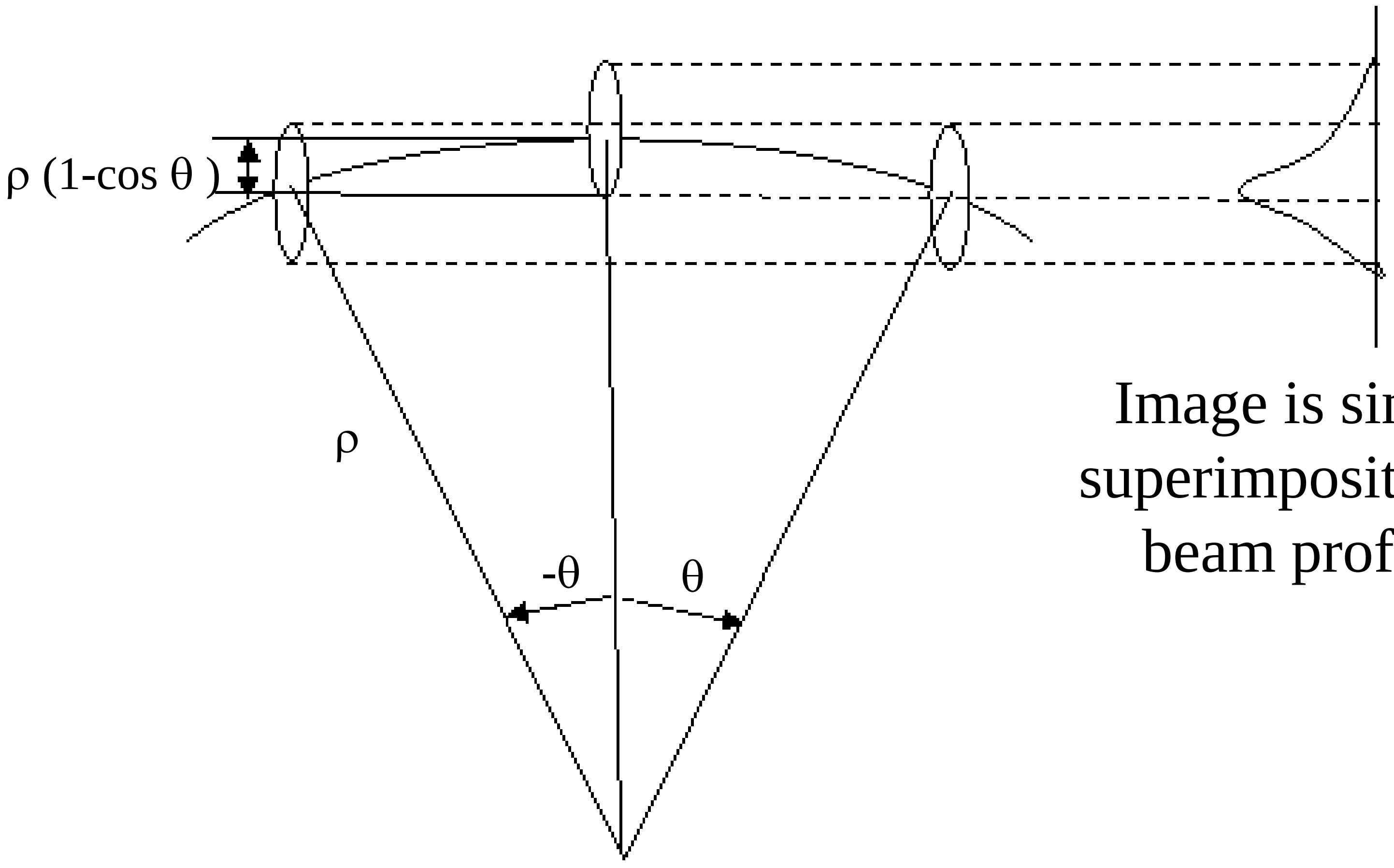


Image is simply superimposition of beam profiles

For example, in the LHC case, $\rho \approx 6000m,\ \theta \approx 0.24mrad,\ \ \rho(1-\cos\theta) \approx 0.173mm$

Each light source has individual Gaussian profile as;

$$\frac{\exp\left[\dfrac{-\left[x-\rho\left\{1-\cos(\theta)\right\}\right]^2}{2\sigma^2}\right]}{\sigma\cdot\sqrt{2\pi}}$$

Where s denotes beam size and term means displacement of center of Gaussian distribution represented by r and q . q is rotation angle of optical axis of pencil of light.

Then apparent beam profile is integrate this by θ

$$\int\frac{\exp\left[\dfrac{-\left[x-\rho\left\{1-\cos(\theta)\right\}\right]^2}{2\sigma^2}\right]}{\sigma\cdot\sqrt{2\pi}}d\theta$$

Then the real part of spatial coherence by this apparent distribution is given by ,

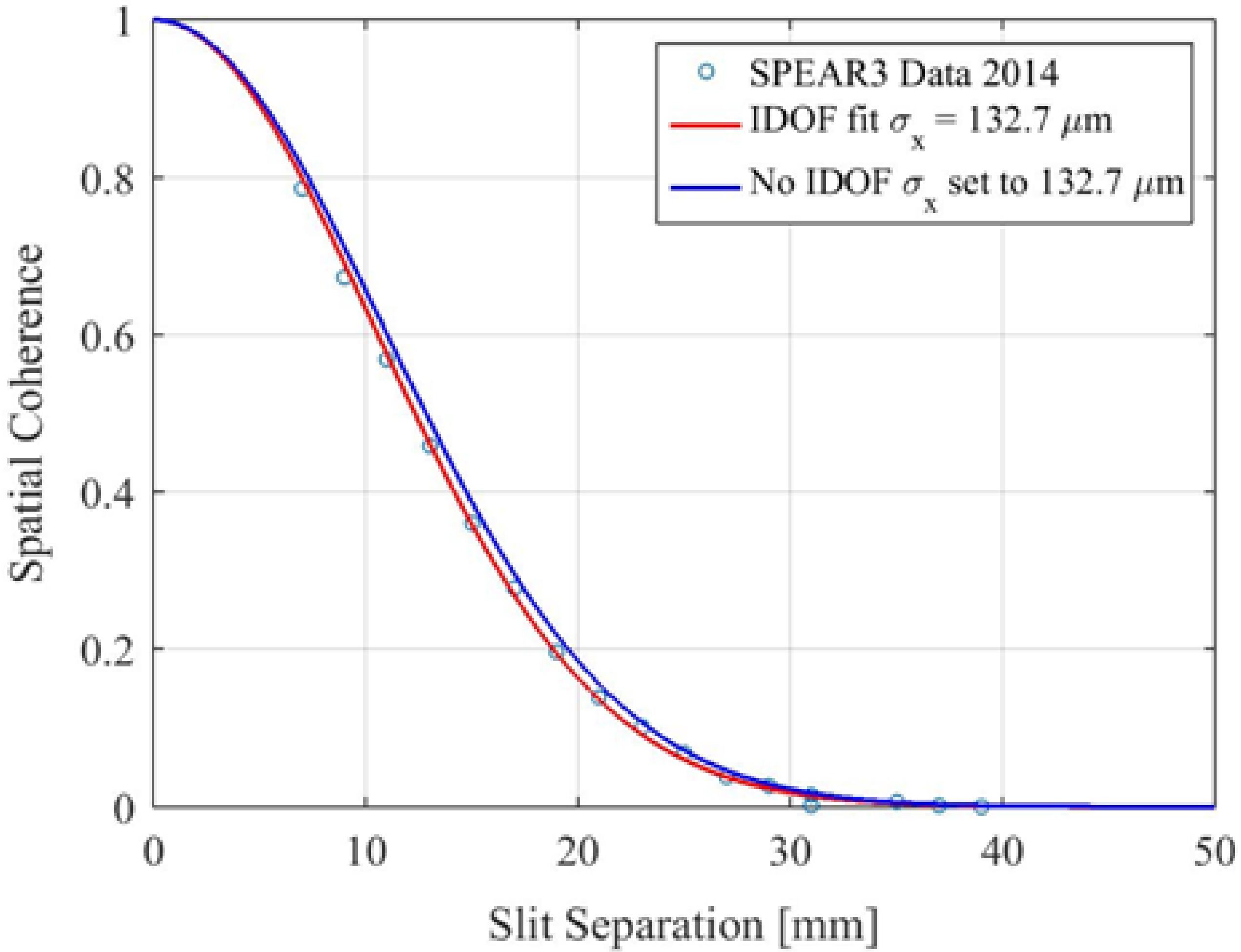
$$\gamma_h(D) = \iint \frac{2\sqrt{I\left(\theta + \frac{D}{2R}\right) I\left(\theta - \frac{D}{2R}\right)}}{I\left(\theta + \frac{D}{2R}\right) + I\left(\theta - \frac{D}{2R}\right)} \cdot I(\theta) \cdot \frac{\exp\left[\frac{-\left[x - \rho\left\{1 - \cos(\theta)\right\}\right]^2}{2\sigma^2}\right]}{\sigma \cdot \sqrt{2\pi}} \cdot \cos\left(\frac{2\pi \cdot D \cdot x}{R \cdot \lambda}\right) d\theta\, dx$$

in here the term

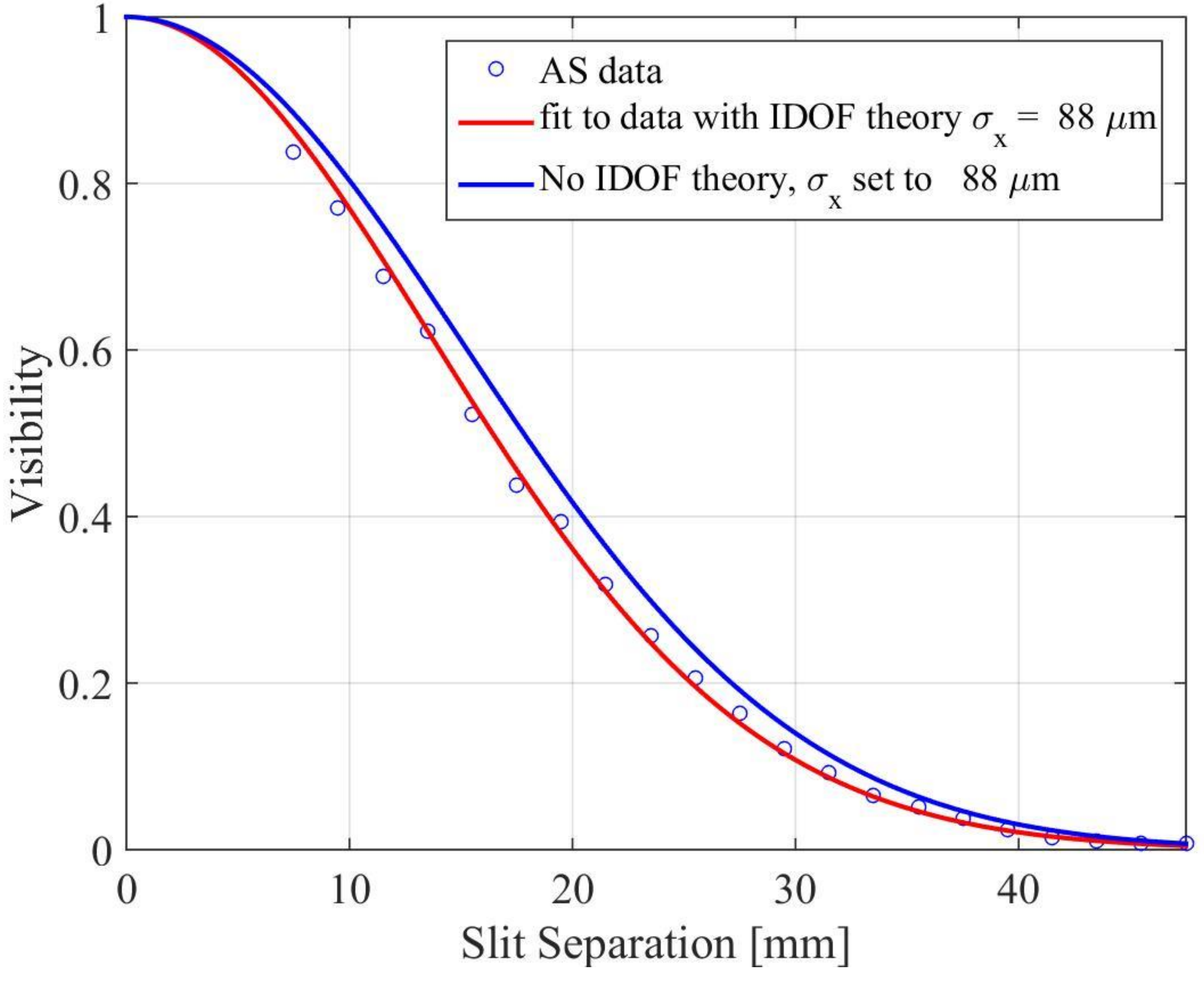
$$\frac{2\sqrt{I\left(\theta + \frac{D}{2R}\right) I\left(\theta - \frac{D}{2R}\right)}}{I\left(\theta + \frac{D}{2R}\right) + I\left(\theta - \frac{D}{2R}\right)} \cdot I(\theta)$$

denotes intensity imbalance at each slit of double slit, and $I(\theta)$ is horizontal normalized intensity distribution of SR

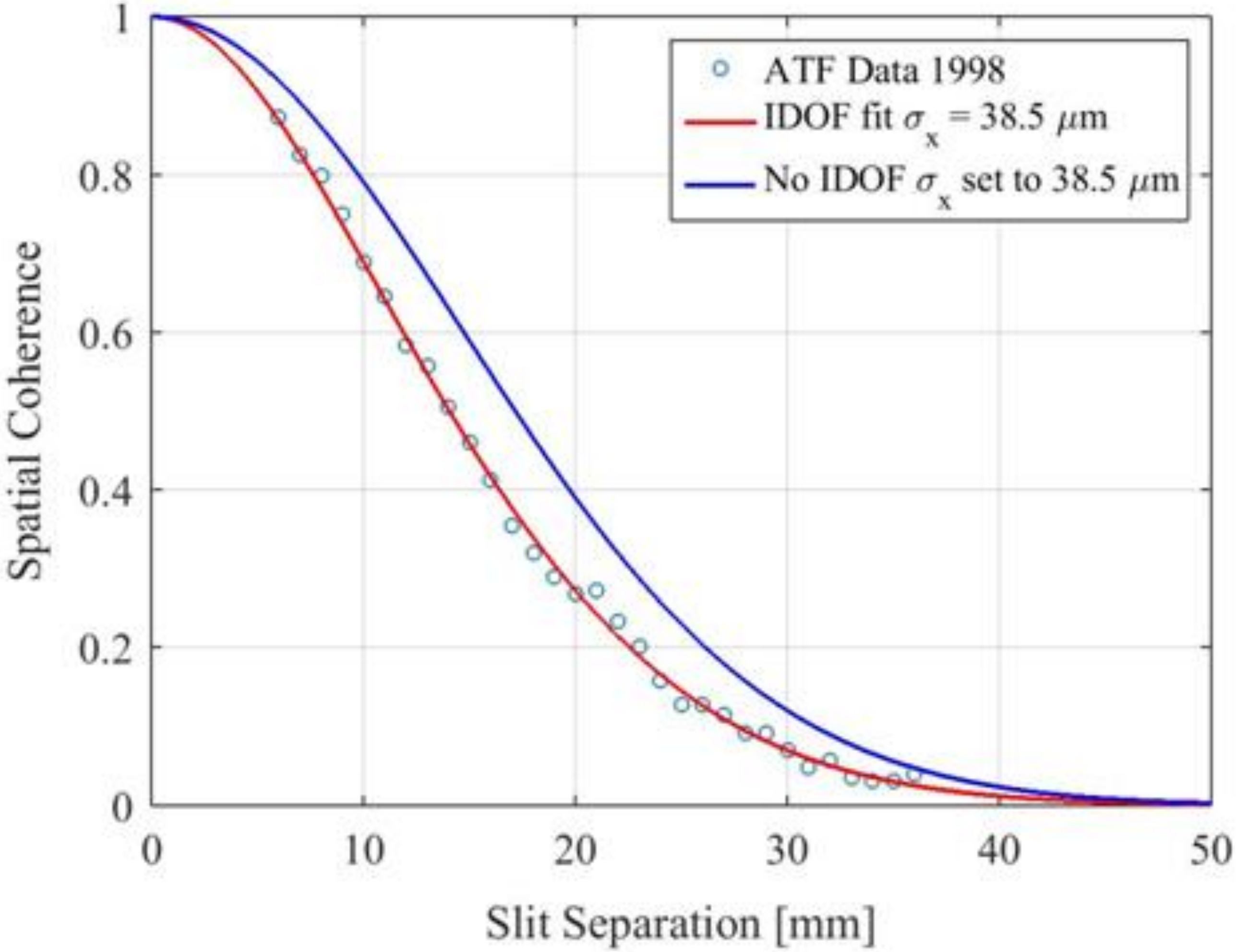
Horizontal beam visibility measurements at SPEAR3. Dots are measurements, red line is fitting with IODF and blue line is calculated visibility curve for a beam size 132.7mm *without* IDOF.



Horizontal visibility measured at ASLS as a function of slit separation. Blue line is calculated visibility curve for a beam size 88μm *without* IDOF.



Horizontal visibility measured at ATF as a function of slit separation. Blue line is calculated visibility curve for a beam size 38.5μm *without* IDOF.



Result

Measured horizontal spatial coherence at 535nm is good agreement with 86μm beam size take into account of incoherent field depth.

Conclusion

Designed horizontal beam size at the source point is 87μm . Lattice parameters are well corrected in the storage ring of Australian Synchrotron. We can conclude measured beam size 86μm is in good agreement to the designed horizontal beam size.

3-6. Small beam size measurement using Gaussian beam profile approximation at PF, AURORA, ATF

7-1. SMALL BEAM SIZE MEASUREMENT

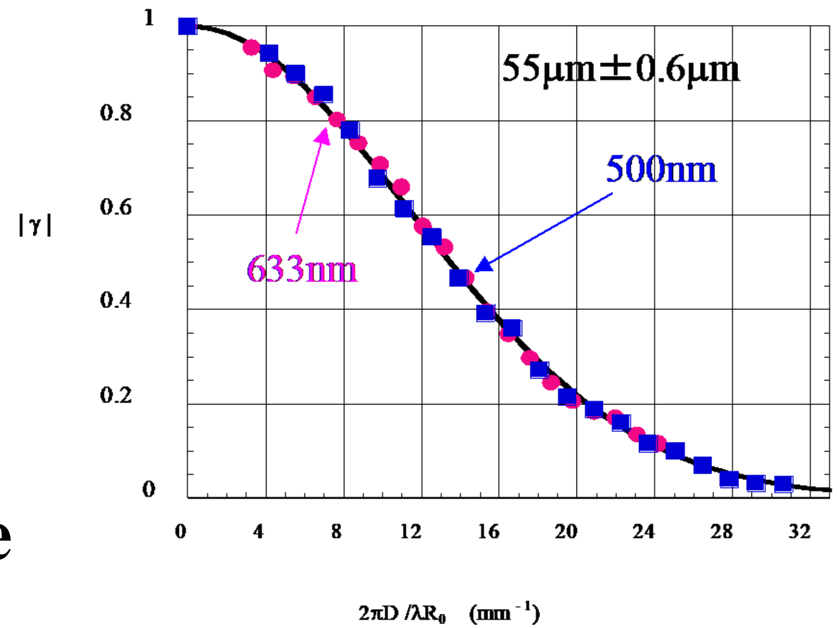
We often approximate the beam profile with a Gaussian shape. A spatial coherence is also given by a Gauss function. We can evaluate a RMS width of spatial coherence by using q least-squares analysis. The RMS beam size σ_{beam} is given by the RMS width of the spatial coherence curve σ_γ as follows:

$$\sigma_{beam} = \frac{\lambda \cdot R}{2 \cdot \pi \cdot \sigma_\gamma}$$

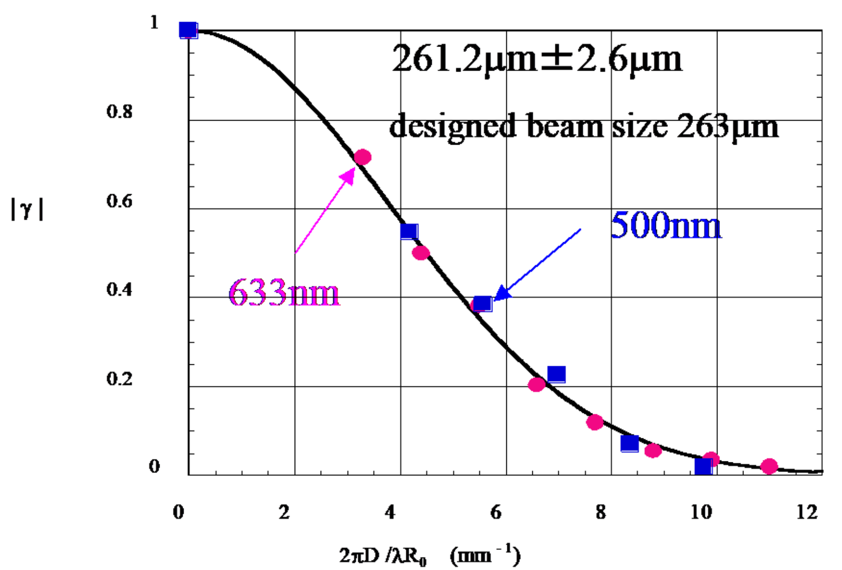
where R : distance between the beam and the double slit.

λ : wave length

7-2. Vertical and horizontal beam size in low emittance lattice at the Photon Factory



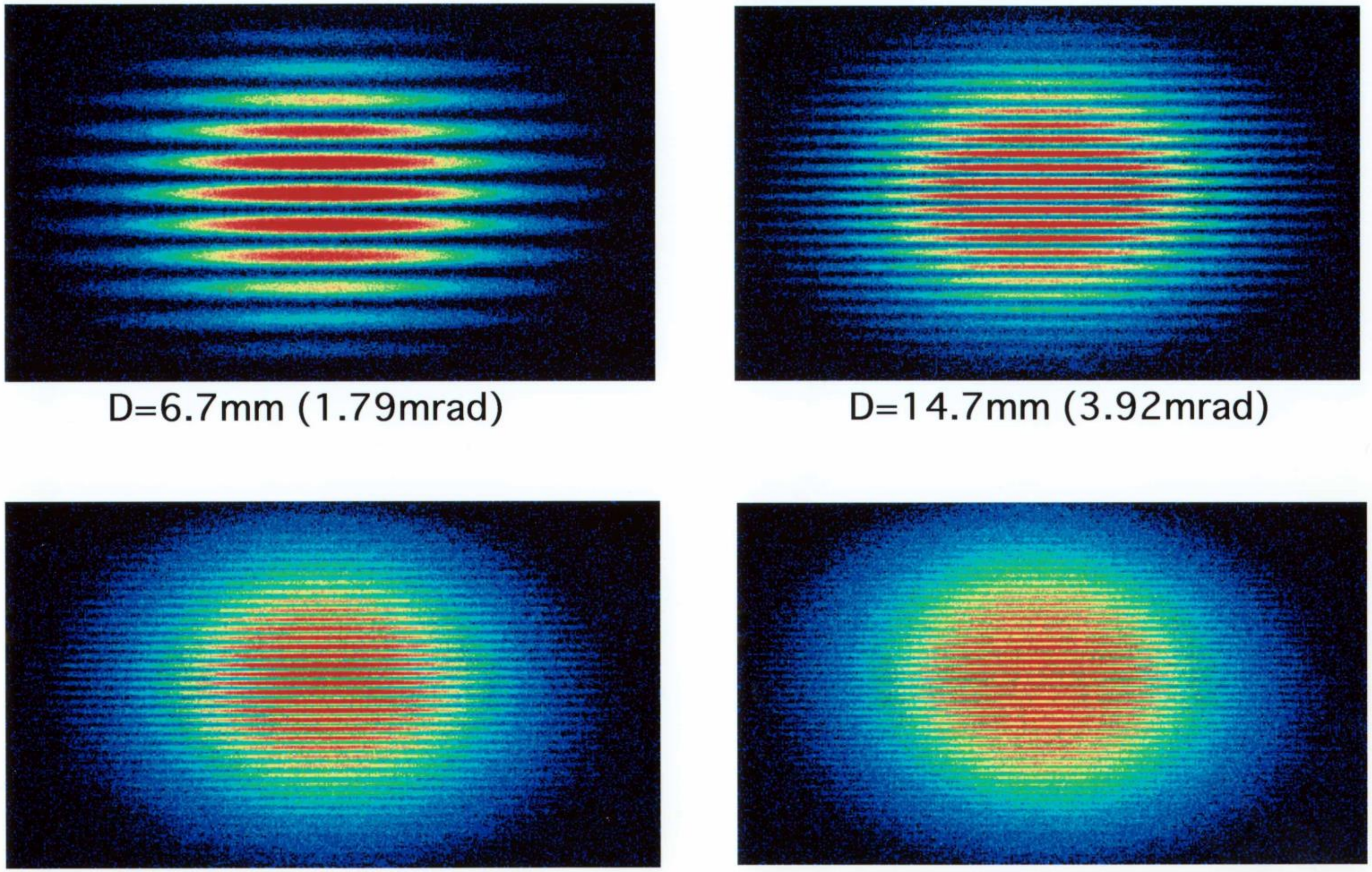
(a) vertical



(b) horizontal

7-3. Vertical beam size at the SR center of Ritsumeikan university AURORA.

λ=550nm

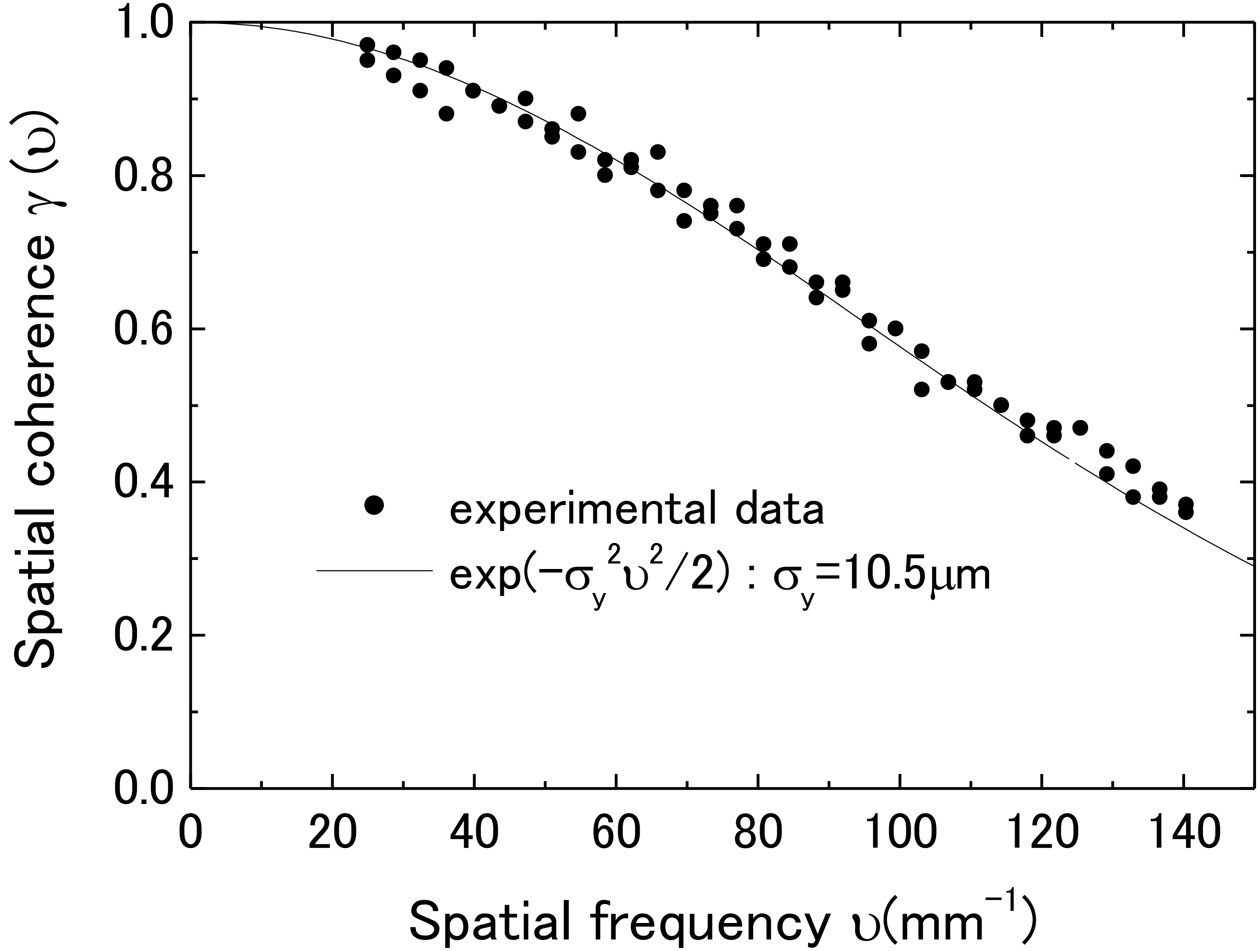


D=6.7mm (1.79mrad)

D=14.7mm (3.92mrad)

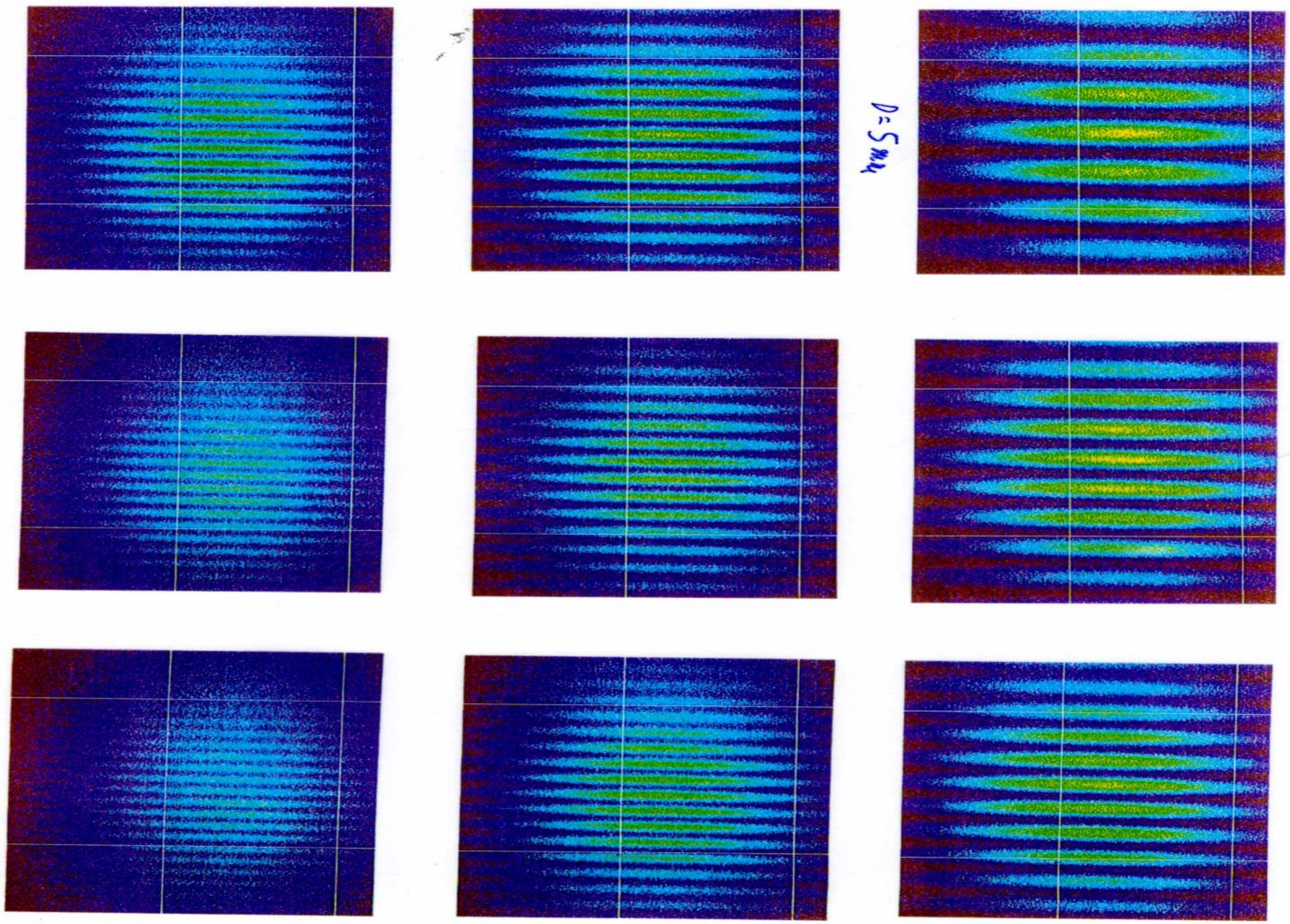
D=22.7mm (6.05mrad)

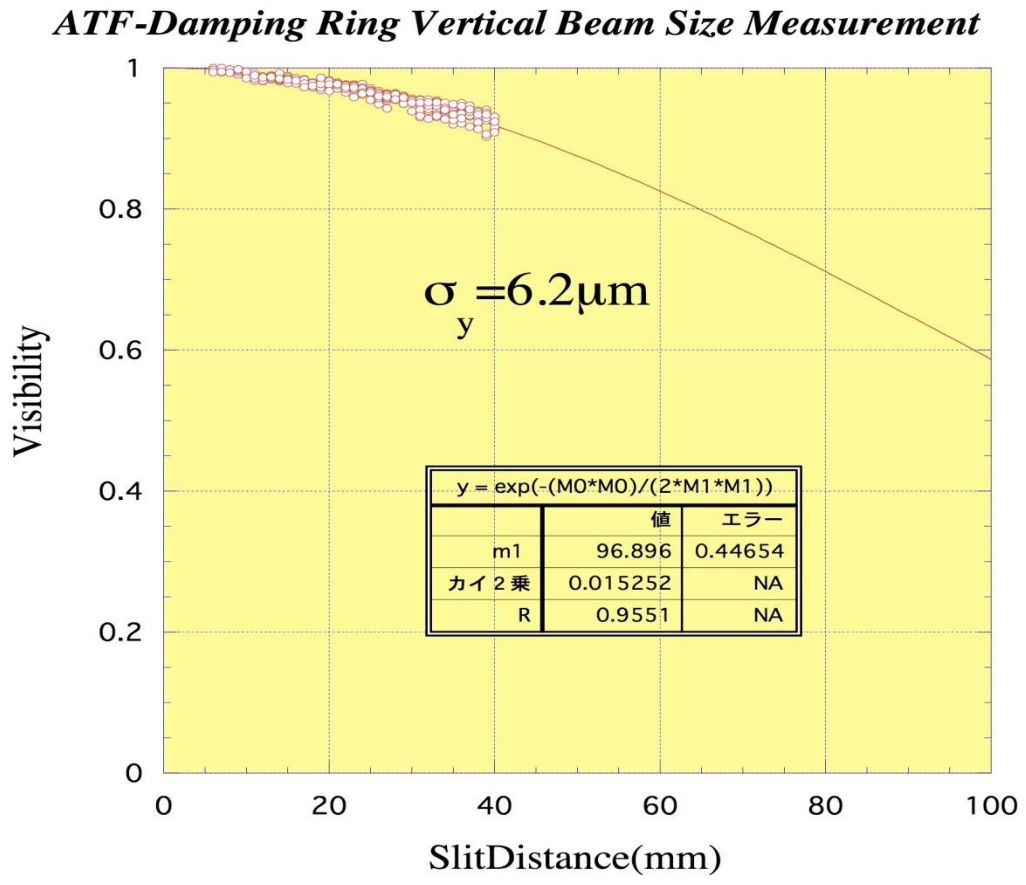
D=28.7mm (7.65mrad)



Spatial coherence γ (υ)
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
120
140
Spatial frequency υ(mm^{-1})
experimental data
$\exp(-\sigma_y^2\upsilon^2/2)$: σ_y=10.5μm

7-4. Vertical beam size at the ATF,KEK





ATF-Damping Ring Vertical Beam Size Measurement
Visibility
SlitDistance(mm)
0
0.2
0.4
0.6
0.8
1
20
40
60
80
100
σ_y=6.2μm
y = exp(-(M0*M0)/(2*M1*M1))
値
エラー
m1
96.896
0.44654
カイ２乗
0.015252
NA
R
0.9551
NA

3-7. SR interferometer as a daily beam size monitor

We can also evaluate the RMS. beam size from one data of visibility, which is measured at a fixed separation of double slit. The RMS beam size σ_{beam} is given by ,

$$\sigma_{beam} = \frac{\lambda \cdot R_0}{\pi \cdot D} \cdot \sqrt{\frac{1}{2} \cdot \ln\left(\frac{1}{\gamma}\right)}$$

where γ denotes the visibility, which is measured at a double slit separation of D.

To consider that in the case to make an image, the resolution is limited by diffraction which is a Fourier transform using a given region of spatial frequency space (measurement in the real space).

In the case of interferometry, we can measure a small beam size with limited region of spatial frequency space by means of these two methods (measurement in the inverse space).

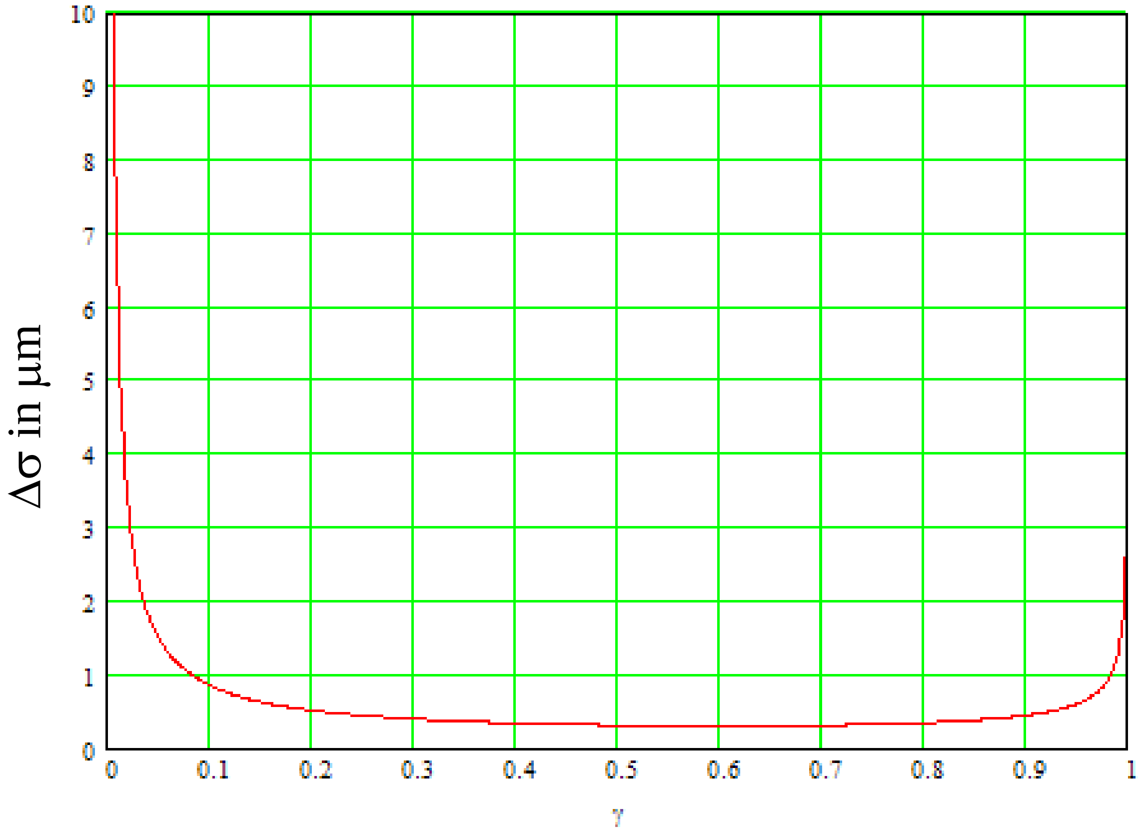
Error transfer
from $\Delta\gamma$ to $\Delta\sigma$

with constant $\Delta\gamma$

$$\Delta\sigma \propto \frac{1}{\gamma \cdot \ln\left(\frac{1}{\gamma}\right)} \Delta\gamma$$

Let assume we measured γ with 1% error, and use a typical conditions for wavelength, distance and separation of double slit.

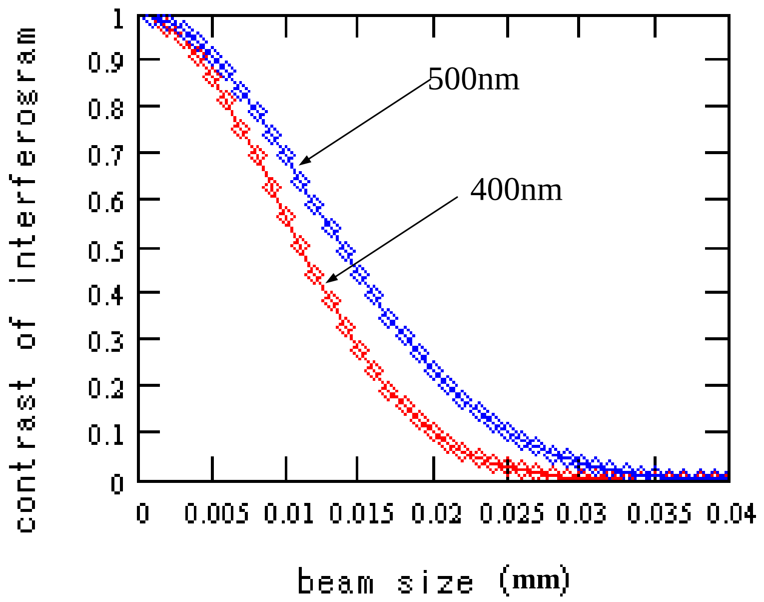
To allow error in beam size 1μm, we can measure at γ=0.98



How small beam can we measure?

Assuming to use ATF conditions;

1. 500nm and 400nm
2. double slit separation 50mm 6.7mrad
3. distance between source point and double slit is 7.4m.



Smallest result by this method is 4.7µm with 400nm at the ATF, KEK

With 500nm;

beam size 3.0µm → contrast : 0.97

beam size 4.0µm → contrast : 0.94

If we measure contrast of the interferomgram in 1%, we can measure difference between 3µm and 4µm with a resolution less than 1µm

Automatic beam size measurement for daily monitor

To find visibility from interferogram, we fit the measured interferogram using the standard Levenberg-Marquadt non-linear fitting.

Intensity distribution of interferogram by qusi-nomochromatic ray is given by,

$$I(y,D)=(I_1+I_2)\cdot\int\left\{\mathrm{sinc}\left(\frac{\pi\cdot a\cdot y\cdot\chi(D)}{\lambda\cdot f}\right)\right\}^2\cdot\left\{1+\gamma\cdot\cos\left(k\cdot D\cdot\left(\frac{y}{f}+\psi\right)\right)\right\}d\lambda.$$

The input to the Marquardt algorithm is thus the fitting function,

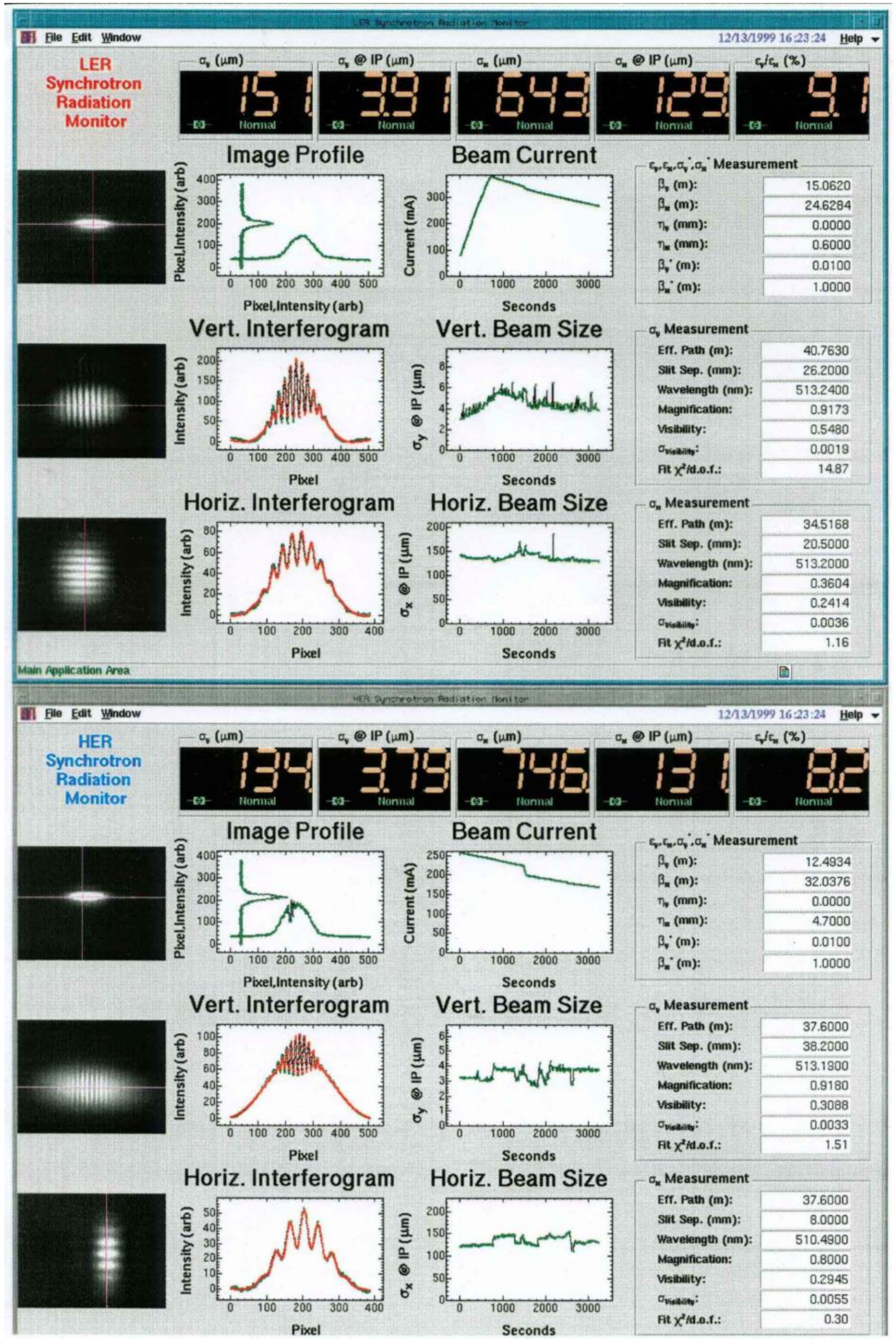
$$I=a_1+a_2y+\sum_{i=1}^{n}f_i(\lambda_i)\cdot\frac{a_3}{2}\cdot\left\{\mathrm{sinc}\left(\frac{\lambda_0}{\lambda_i}\cdot a_4(y-a_5)\right)\right\}^2\cdot\left\{1+a_6\cdot\cos\left(\frac{\lambda_0}{\lambda_i}\cdot a_7(y-a_8)\right)\right\}$$

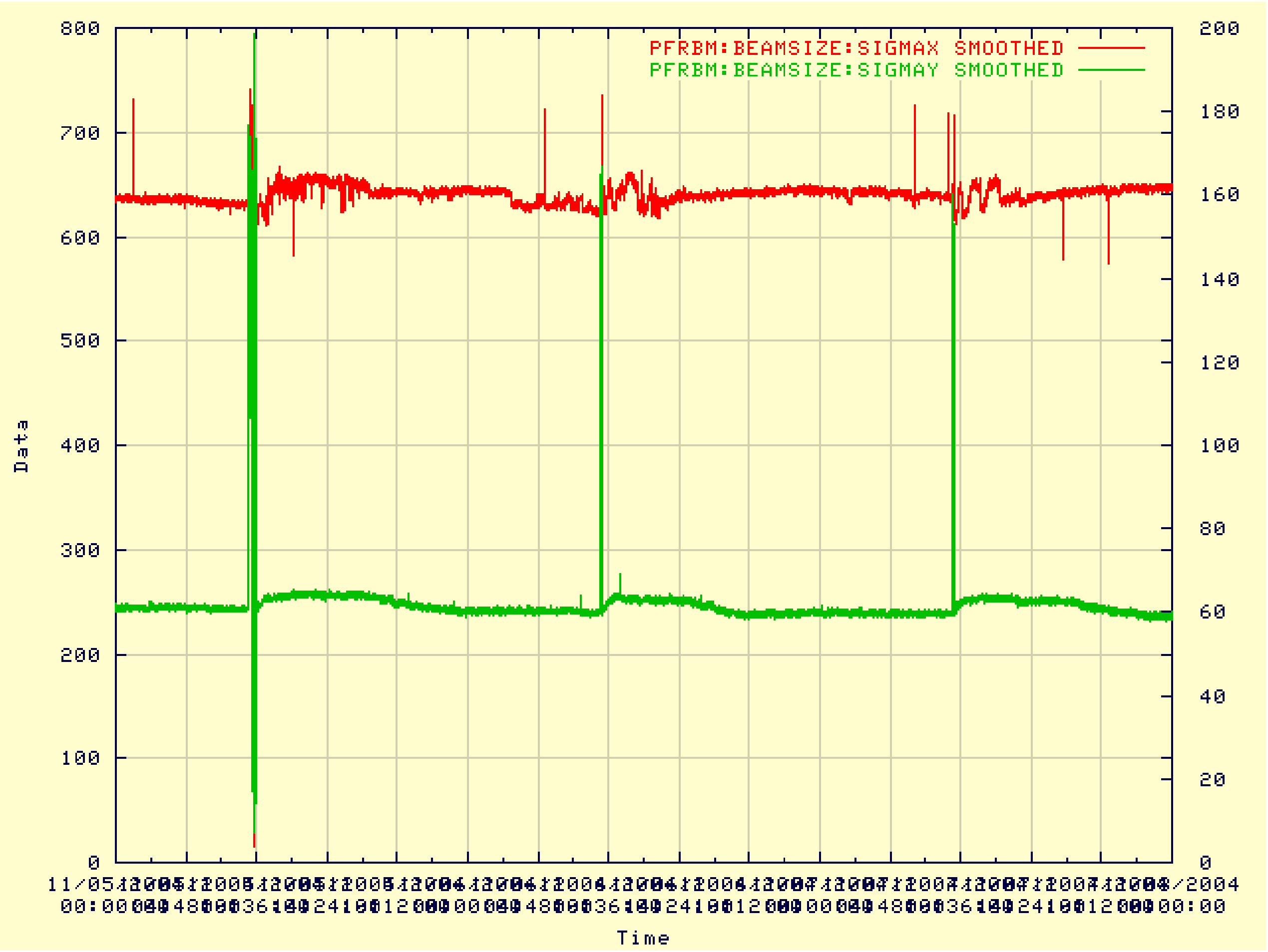
where f_i is rocking function of band-pass filter.

The interferogram is digitized from CCD camera image and analyzed using a commercial image analysis package Wit.

The Wit provides hooks for user-developed routines to be included.
We install the Marquardt fitting procedure in Wit.

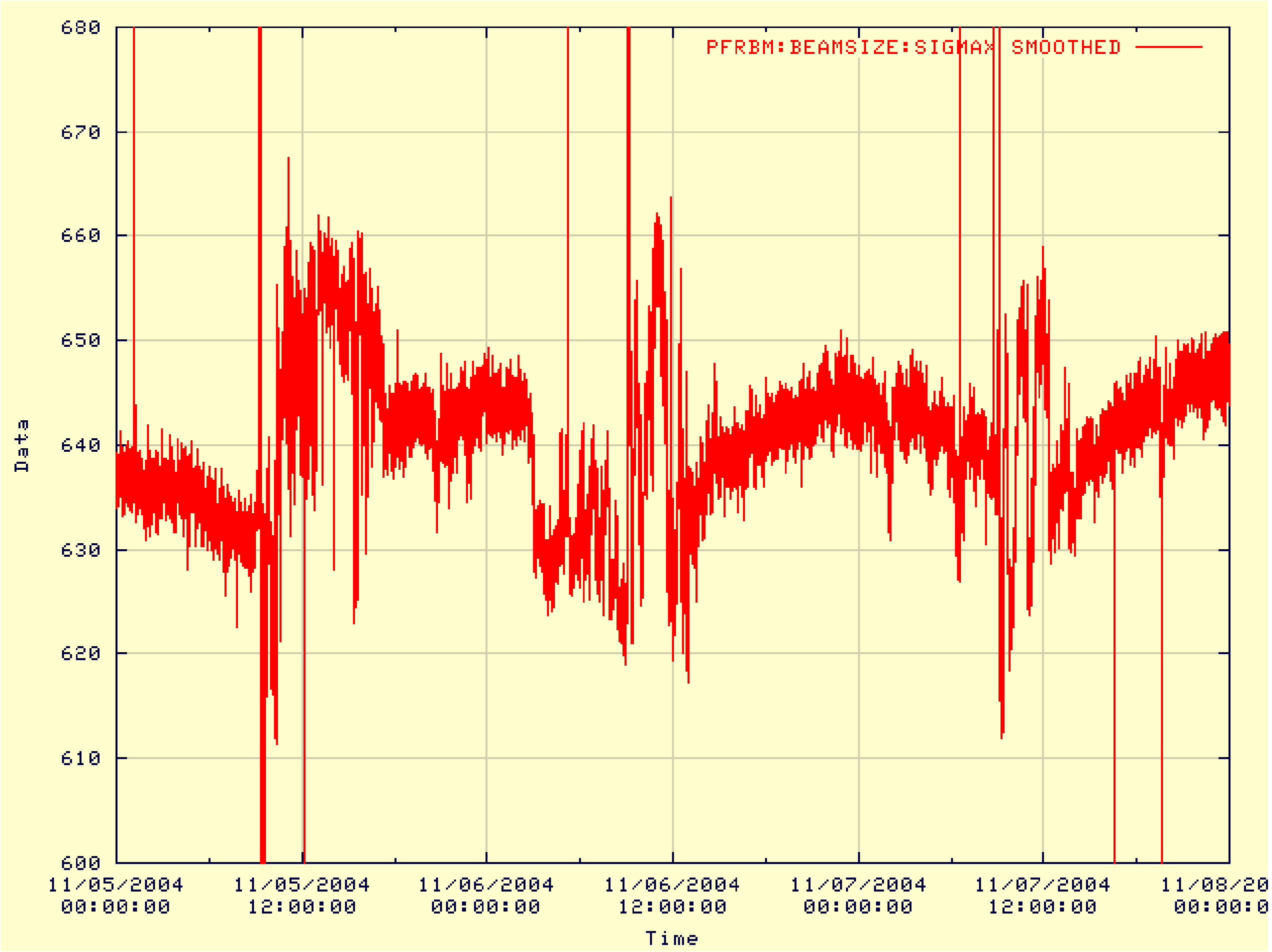
The bulk of the work of image analysis is performed on a computer beside the interferometer.
The results are relayed to a computer in the control room for display and further analysis.


File Edit Window
12/13/1999 16:23:24 Help
LER Synchrotron Radiation Monitor
σy (μm) 151
σy @ IP (μm) 3.91
σx (μm) 643
σx @ IP (μm) 129
εy/εx (%) 9.1
Normal
Image Profile
Pixel,Intensity (arb)
Beam Current
Current (mA)
Seconds
εy,εx,σy*,σx* Measurement
βy (m): 15.0620
βx (m): 24.6284
ηy (mm): 0.0000
ηx (mm): 0.6000
βy* (m): 0.0100
βx* (m): 1.0000
Vert. Interferogram
Intensity (arb)
Pixel
Vert. Beam Size
σy @ IP (μm)
Seconds
σy Measurement
Eff. Path (m): 40.7630
Slit Sep. (mm): 26.2000
Wavelength (nm): 513.2400
Magnification: 0.9173
Visibility: 0.5480
σVisibility: 0.0019
Fit χ2/d.o.f.: 14.87
Horiz. Interferogram
Intensity (arb)
Pixel
Horiz. Beam Size
σx @ IP (μm)
Seconds
σx Measurement
Eff. Path (m): 34.5168
Slit Sep. (mm): 20.5000
Wavelength (nm): 513.2000
Magnification: 0.3604
Visibility: 0.2414
σVisibility: 0.0036
Fit χ2/d.o.f.: 1.16
Main Application Area
File Edit Window
12/13/1999 16:23:24 Help
HER Synchrotron Radiation Monitor
σy (μm) 134
σy @ IP (μm) 3.79
σx (μm) 746
σx @ IP (μm) 131
εy/εx (%) 8.2
Normal
Image Profile
Pixel,Intensity (arb)
Beam Current
Current (mA)
Seconds
εy,εx,σy*,σx* Measurement
βy (m): 12.4934
βx (m): 32.0376
ηy (mm): 0.0000
ηx (mm): 4.7000
βy* (m): 0.0100
βx* (m): 1.0000
Vert. Interferogram
Intensity (arb)
Pixel
Vert. Beam Size
σy @ IP (μm)
Seconds
σy Measurement
Eff. Path (m): 37.6000
Slit Sep. (mm): 38.2000
Wavelength (nm): 513.1900
Magnification: 0.9180
Visibility: 0.3088
σVisibility: 0.0033
Fit χ2/d.o.f.: 1.51
Horiz. Interferogram
Intensity (arb)
Pixel
Horiz. Beam Size
σx @ IP (μm)
Seconds
σx Measurement
Eff. Path (m): 37.6000
Slit Sep. (mm): 8.0000
Wavelength (nm): 510.4900
Magnification: 0.8000
Visibility: 0.2945
σVisibility: 0.0055
Fit χ2/d.o.f.: 0.30



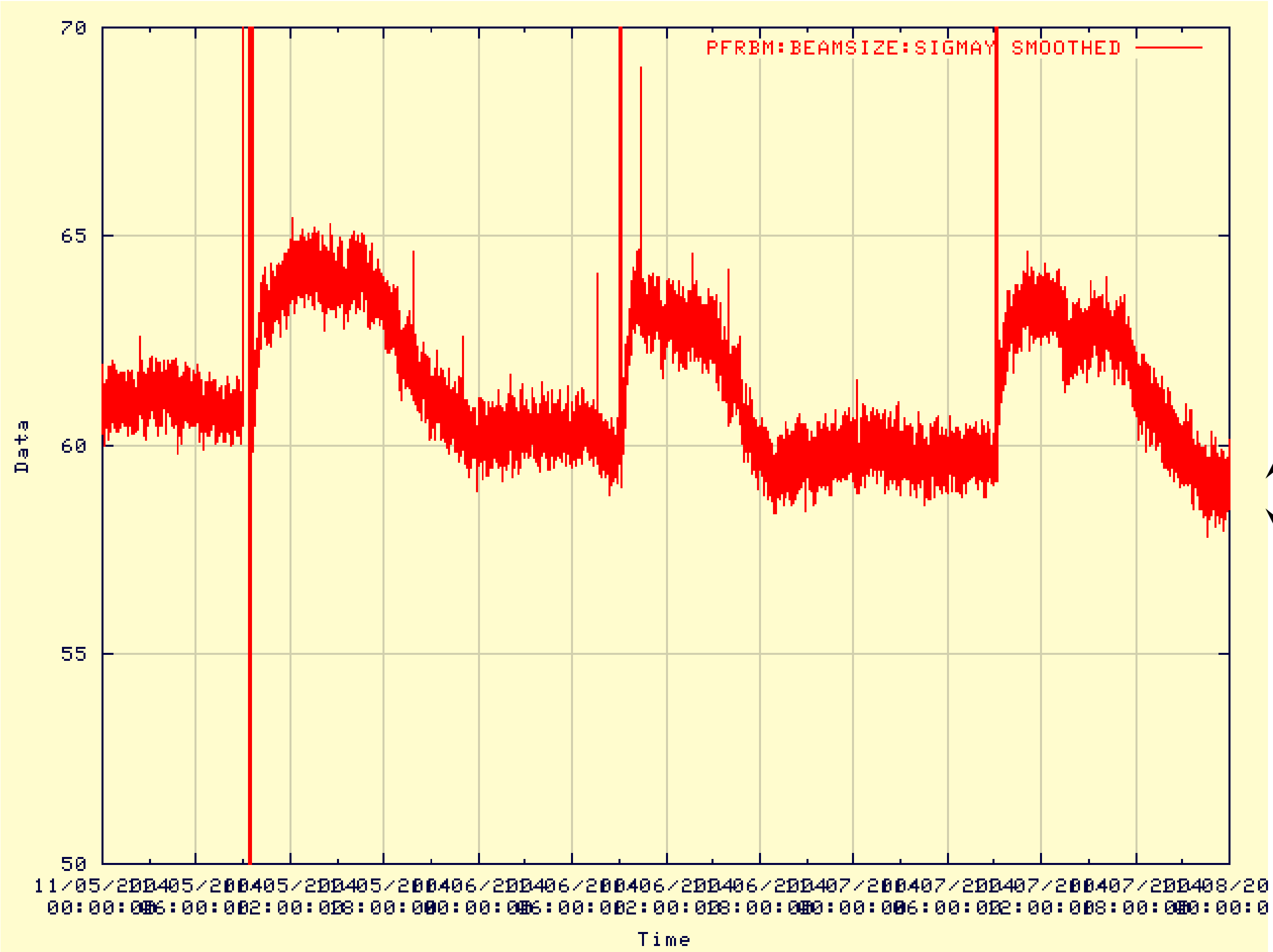
PFRBM:BEAMSIZE:SIGMAX SMOOTHED
PFRBM:BEAMSIZE:SIGMAY SMOOTHED
Data
Time

Horizontal beam size measurement



±3μm

Vertical beam size measurement



±1μm

3-8. Practical notices for using SR interferometer

1) Problems in CCD camera.

linearity, offset, etc.

2) Vibration of building floor.

3) Deformation of mirror.

Calibration of the SR interferometer.

4) Aberrations of optical system of the interferometer including the collimation error.

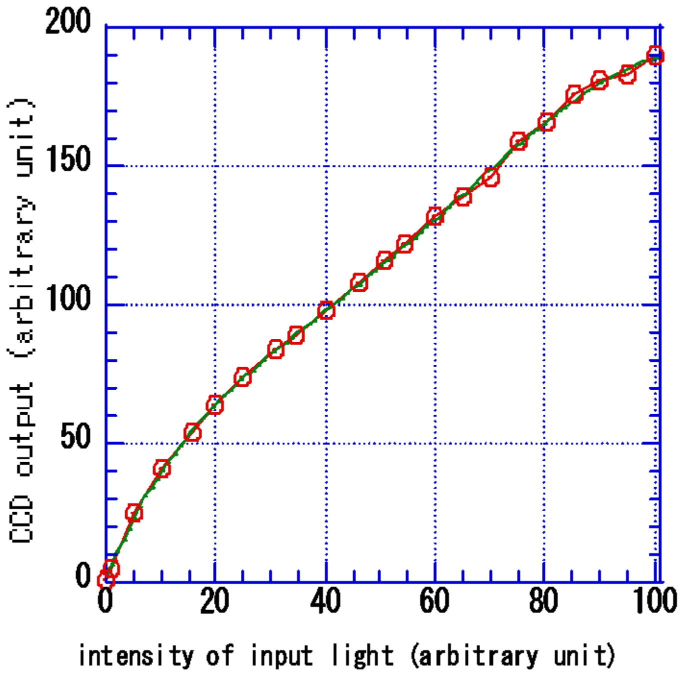
Problems in CCD camera

1. linearity
2. offset of baseline

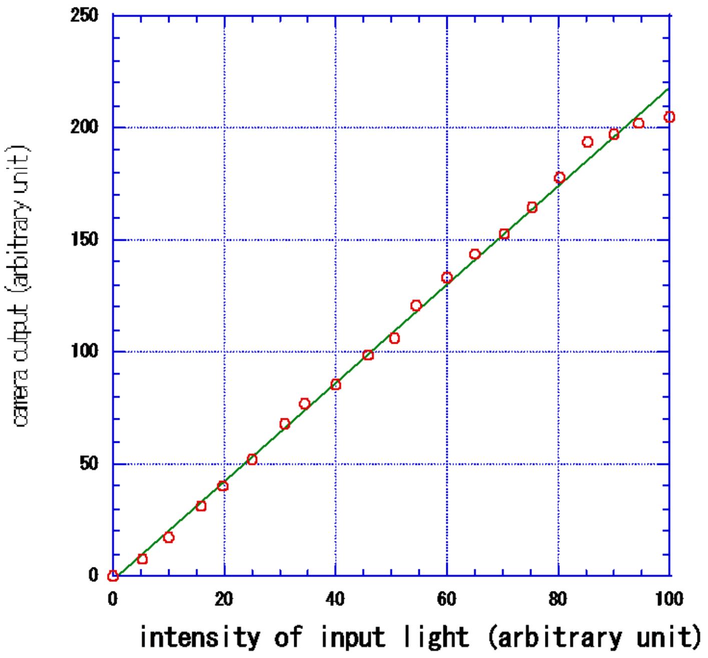
These two problems are especially important for small-beam size measuremnt, because of its interferogram has very high contrast.

These two problems are checked by accurately-calibrated ND filter (calibrated by spectrodensitometer by 0.1%).

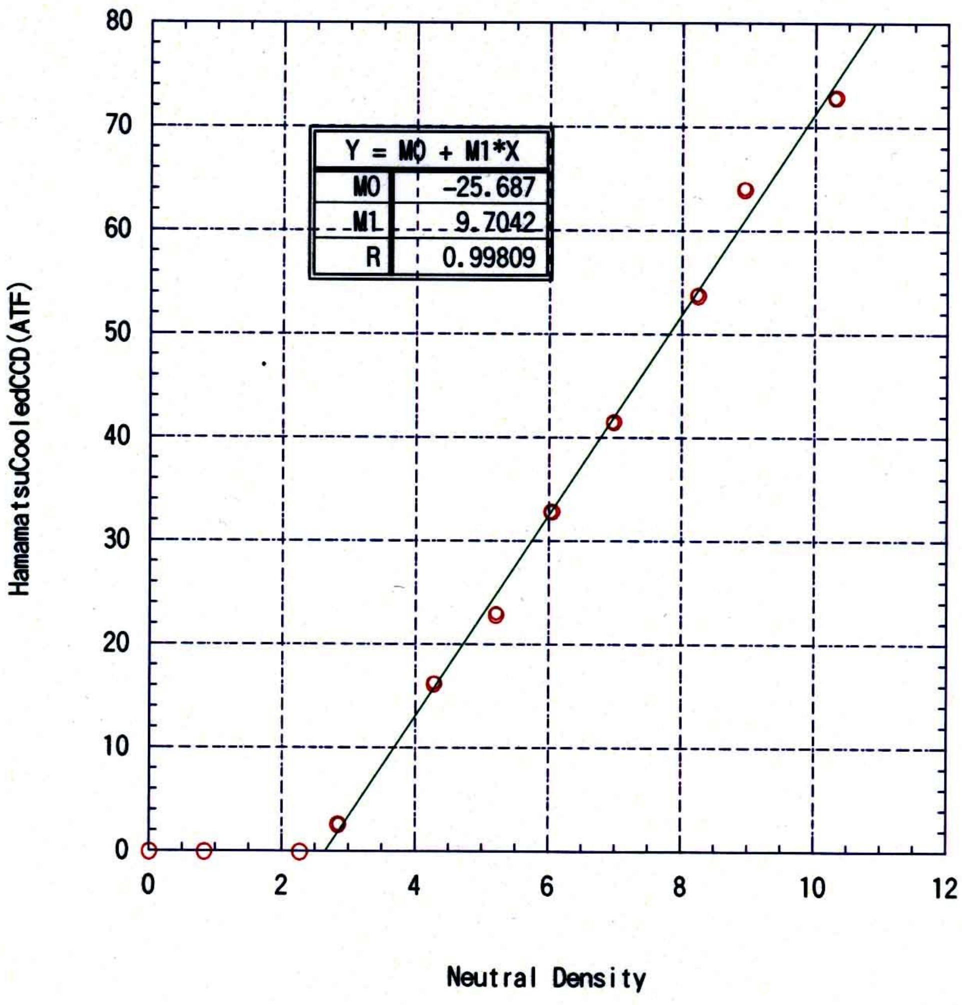
A result of linearity measurement of one CCD cameraused in the SR interferometer.



A result of linearit measurement of linear CCD camera used in the SR interferometer.

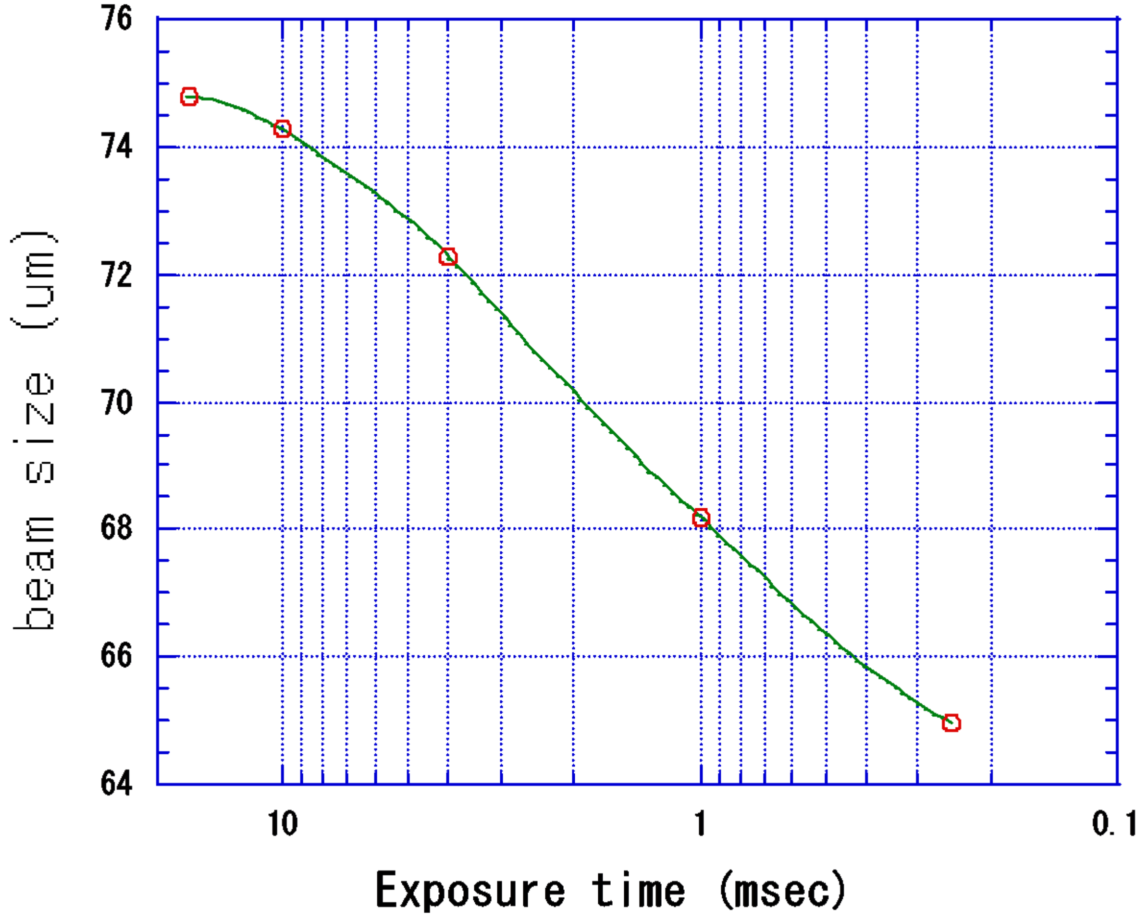


Negative offset of CCD camera



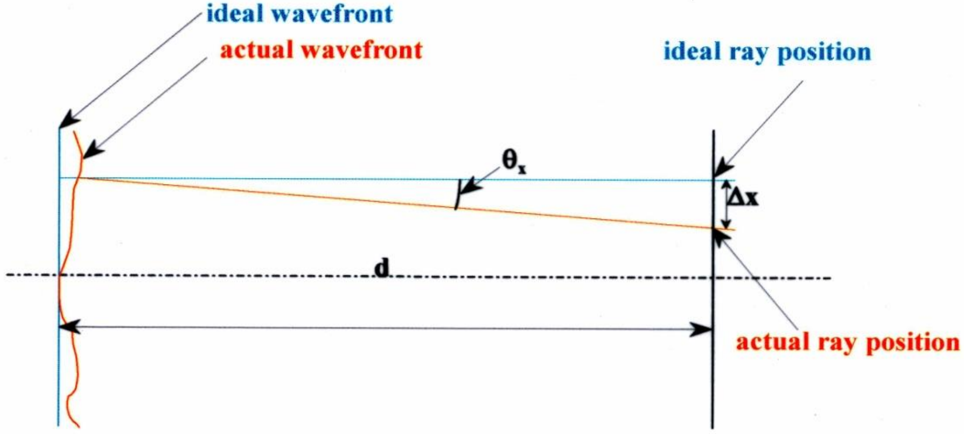
Decrease of observed beam size vs. exposure time of the CCD camera.

Problem of floor vibration



Waveflont distortion sensing by hole array screen (Hartmann screen test)

Measurement of wavefront distortion caused by a deformation of extraction mirror based on Hartmann screen test and its application for calibration of SR interferormter



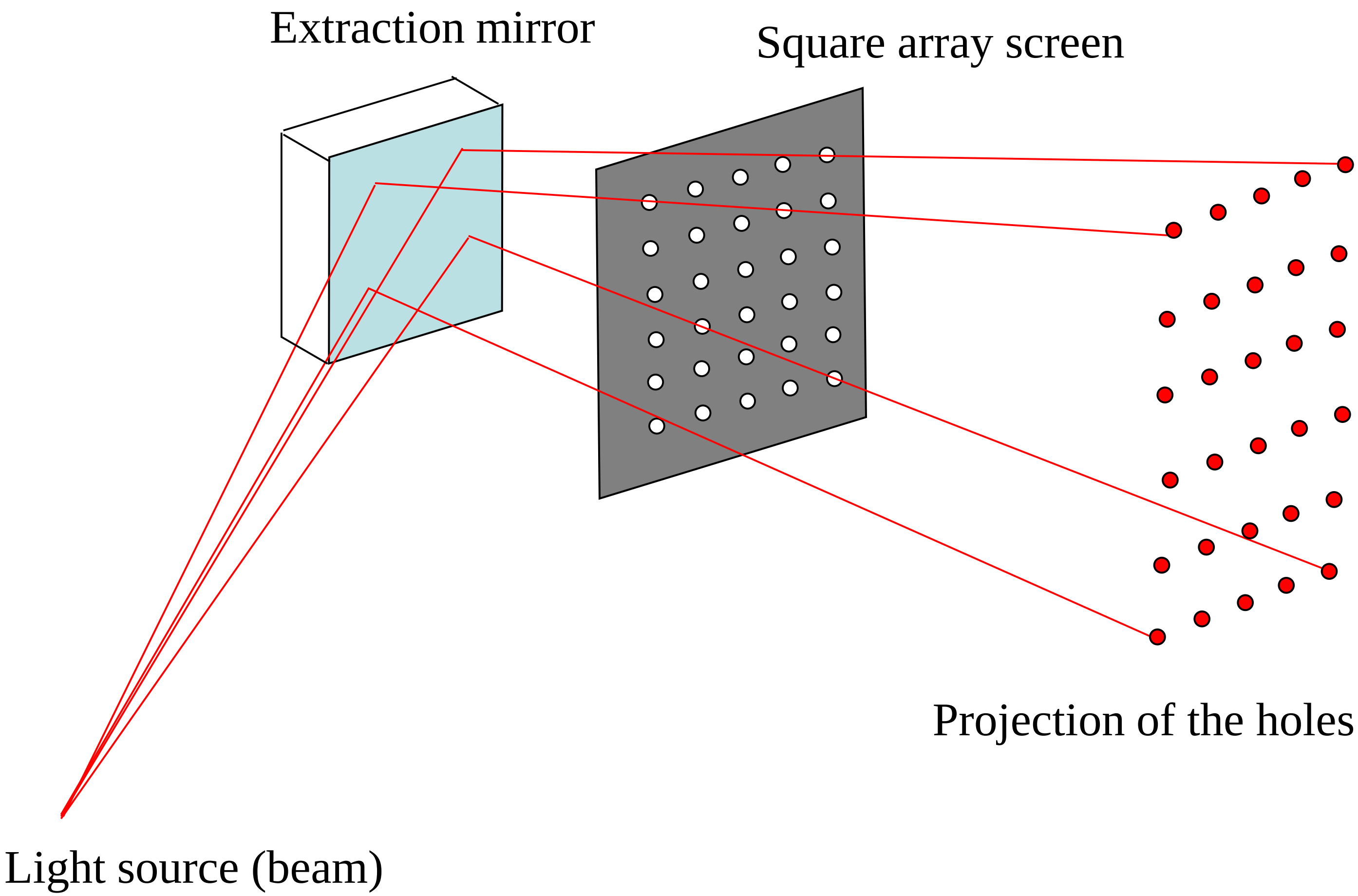
Schemtics of rays and wavefronts in Hartmann test.

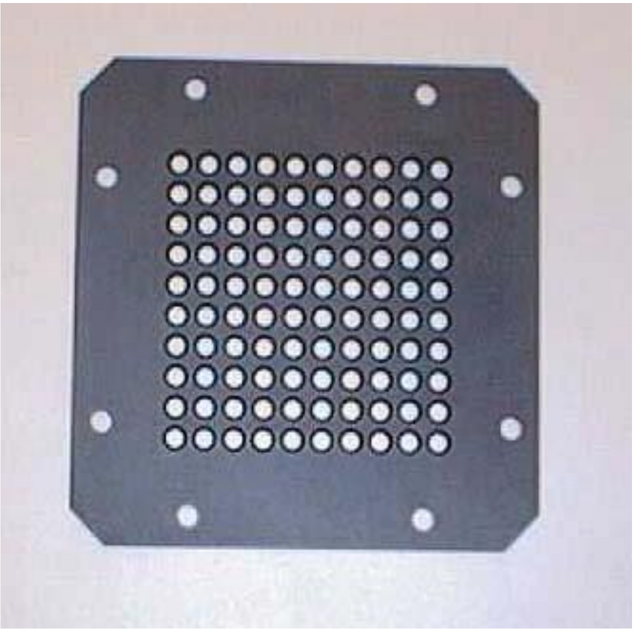
The relation between the wavefront aberration W (separation between ideal wavefront and actual wavefront) and x component Δx of the ray deviation in the observation plane is given by:

$$\frac{\partial W}{\partial x} = \frac{\Delta x}{d} \qquad (1).$$

Integrating this equation, we obtaine

$$W = \frac{1}{d}\int_0^x \Delta x \, dx \qquad (2).$$

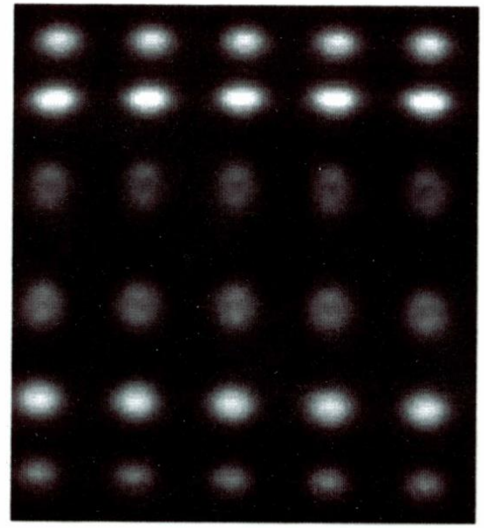

Extraction mirror
Square array screen
Projection of the holes
Light source (beam)



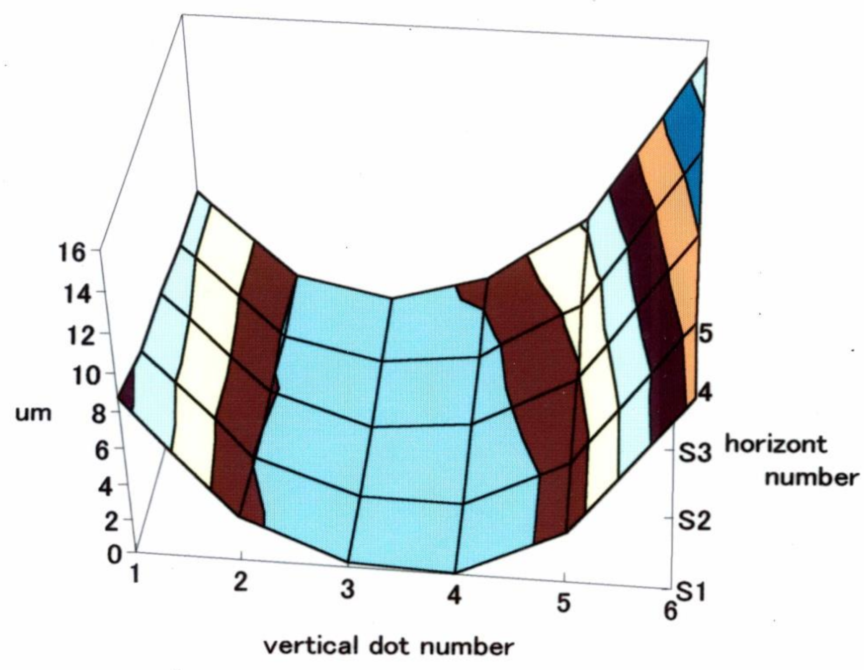
We use a 100-hole square-array screen.

The hole diameter should be small but not so small that their diffraction pattern on the observation plane overlap each other and of sufficient small size to permit an accurate measurement of their positions.

The interval of the holes should be also small having a sufficient number of the sampling points of wave front, but not so small that their interference fringe pattern between surrounding holes stands out.

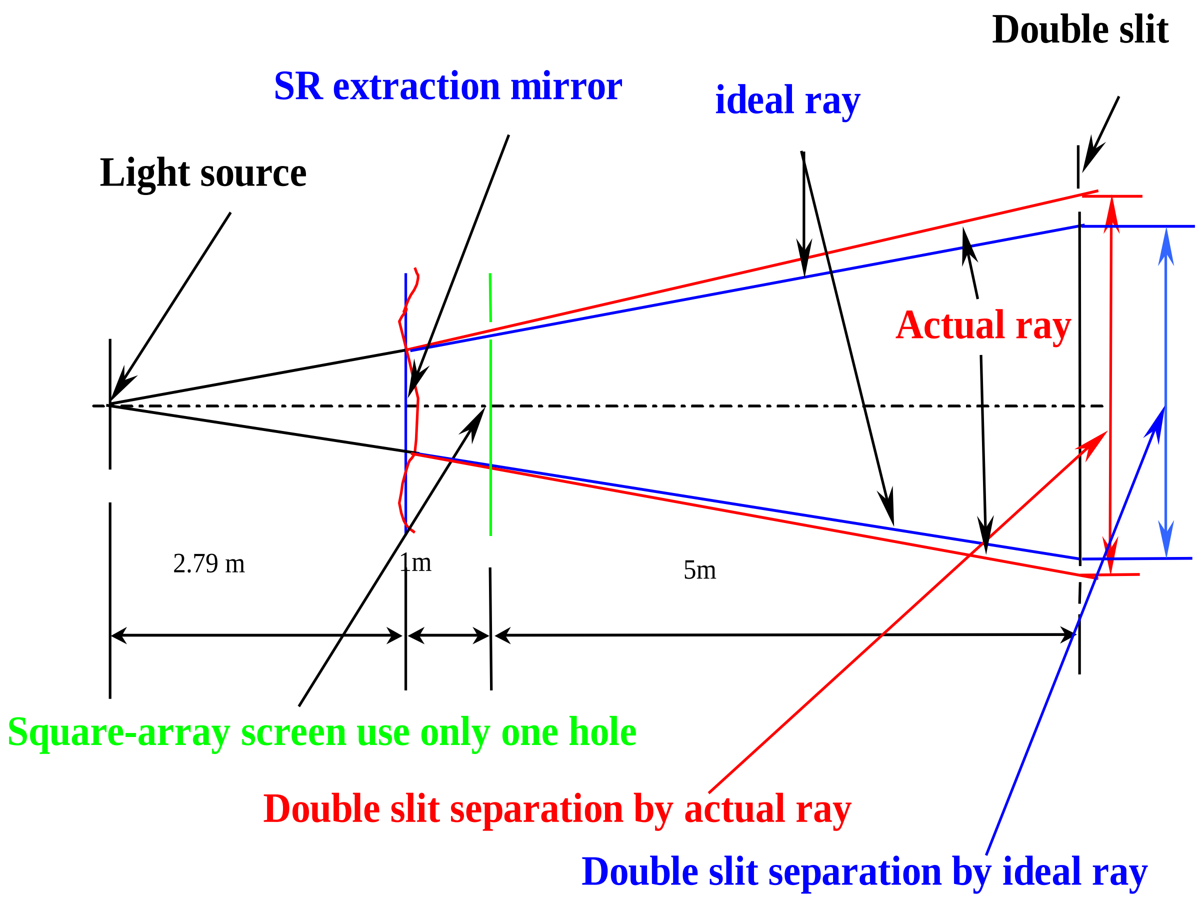


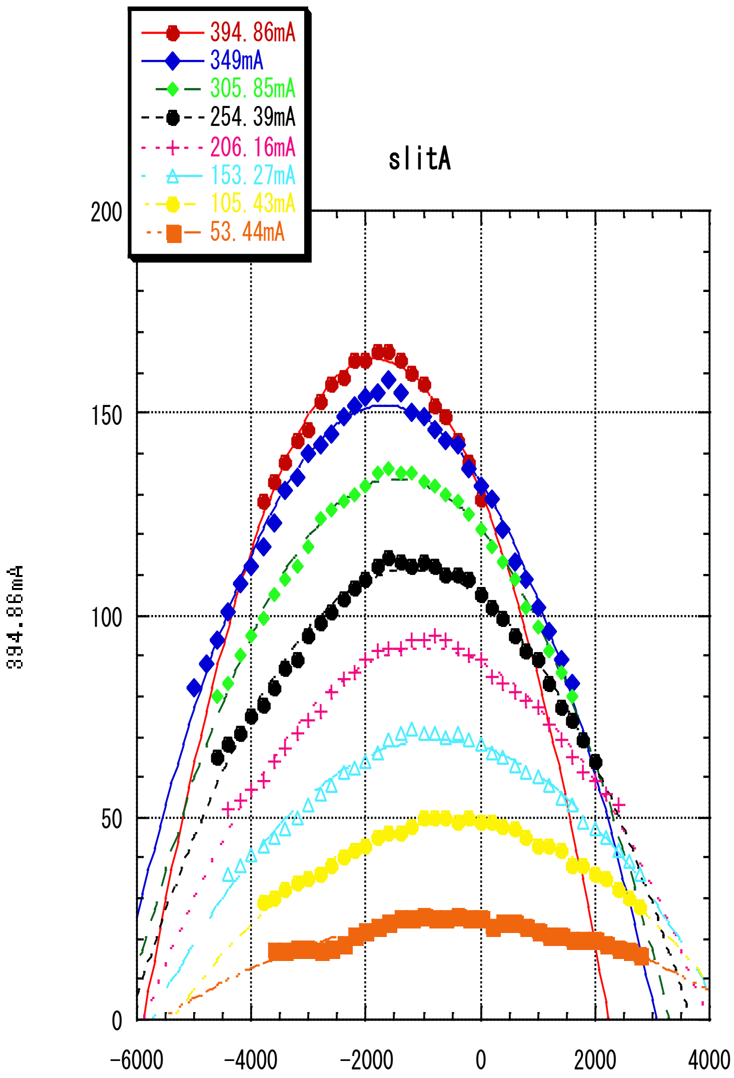
422mA



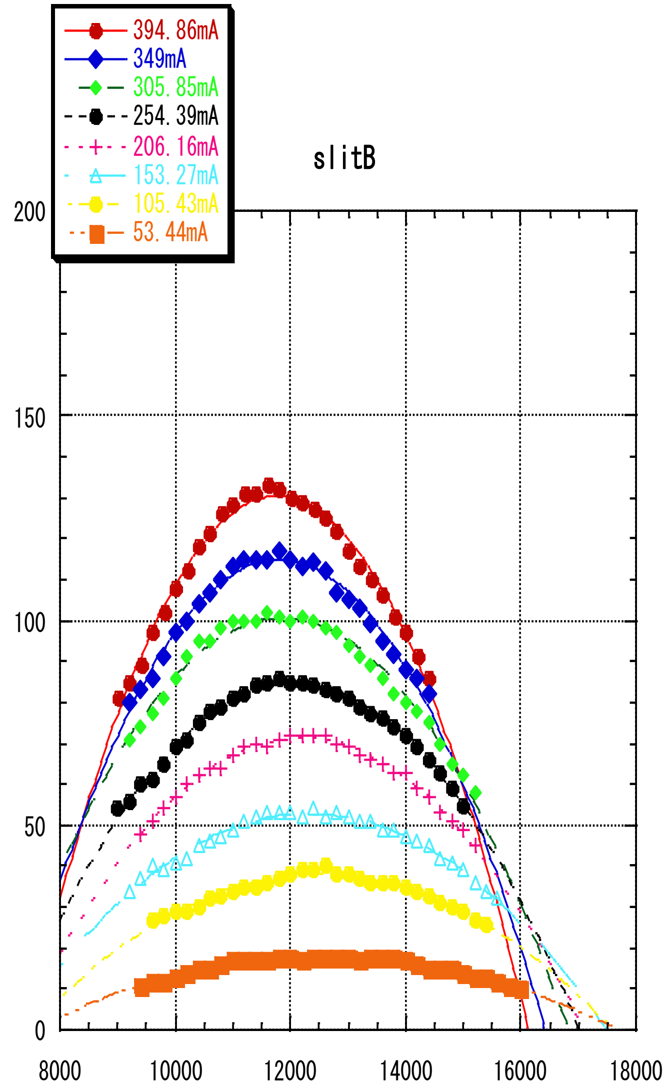
um
16
14
12
10
8
6
4
2
0
1
2
3
4
5
6
vertical dot number
S1
S2
S3
4
5
horizont
number

Calibration of interferometer by a single hole screen (to identify the ideal ray by tracing the actual ray)

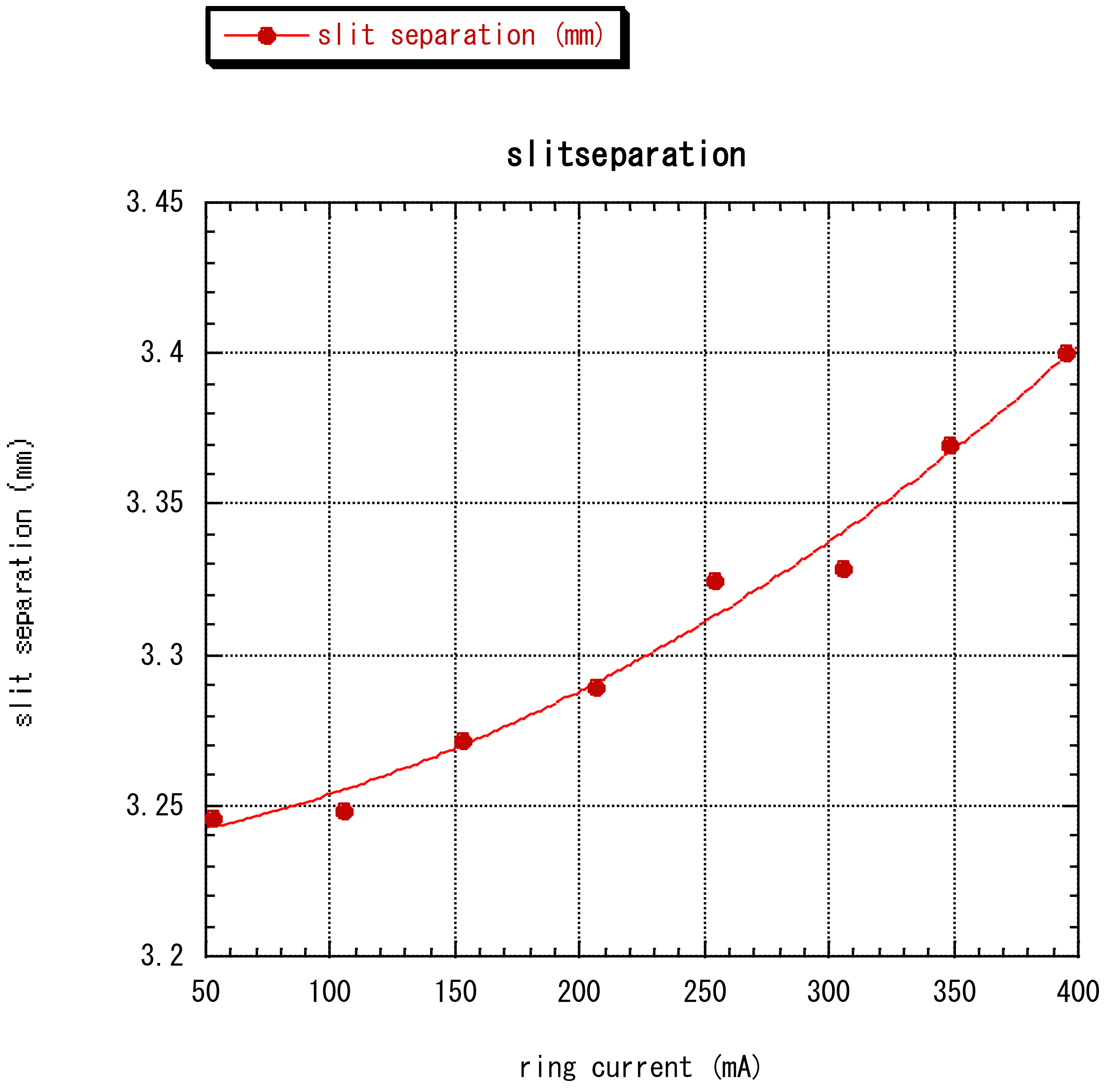




394.86mA
394.86mA
349mA
305.85mA
254.39mA
206.16mA
153.27mA
105.43mA
53.44mA
slitA
200
150
100
50
0
-6000
-4000
-2000
0
2000
4000

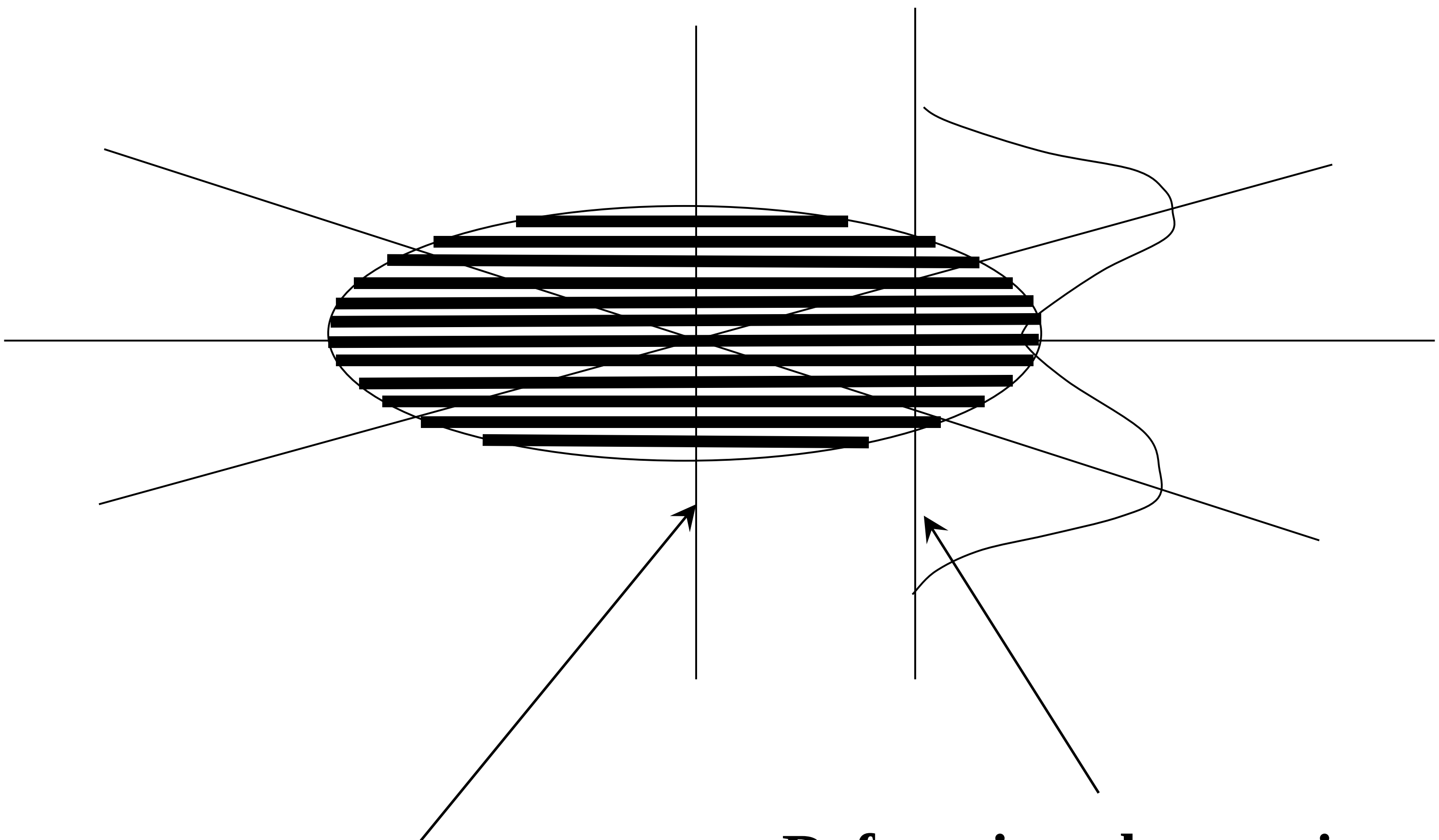


394.86mA
349mA
305.85mA
254.39mA
206.16mA
153.27mA
105.43mA
53.44mA
slitB
200
150
100
50
0
8000
10000
12000
14000
16000
18000



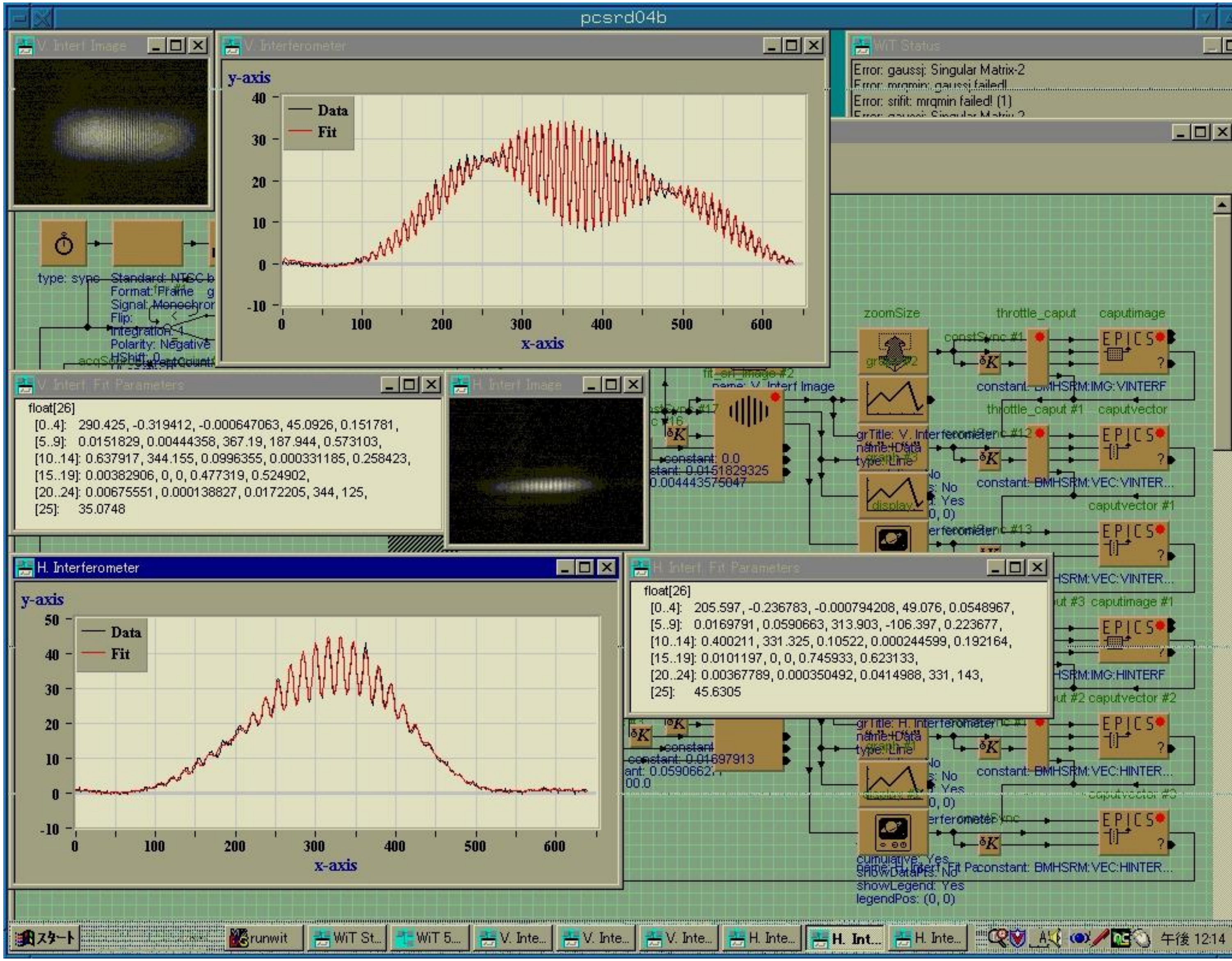
slit separation (mm)
slitseparation
3.45
3.4
3.35
3.3
3.25
3.2
slit separation (mm)
50
100
150
200
250
300
350
400
ring current (mA)

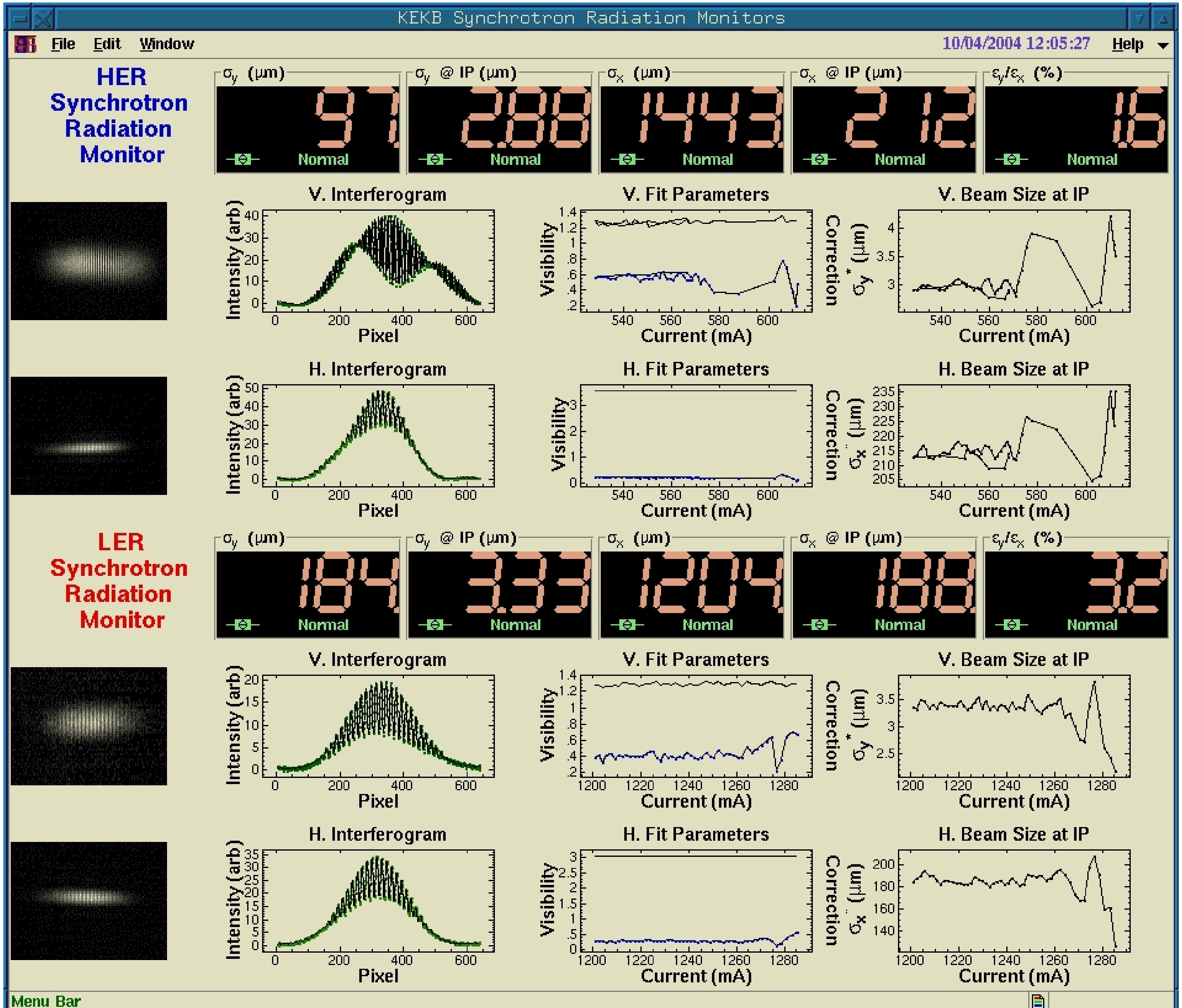
Calibration of SR interferometer with defocusing mode



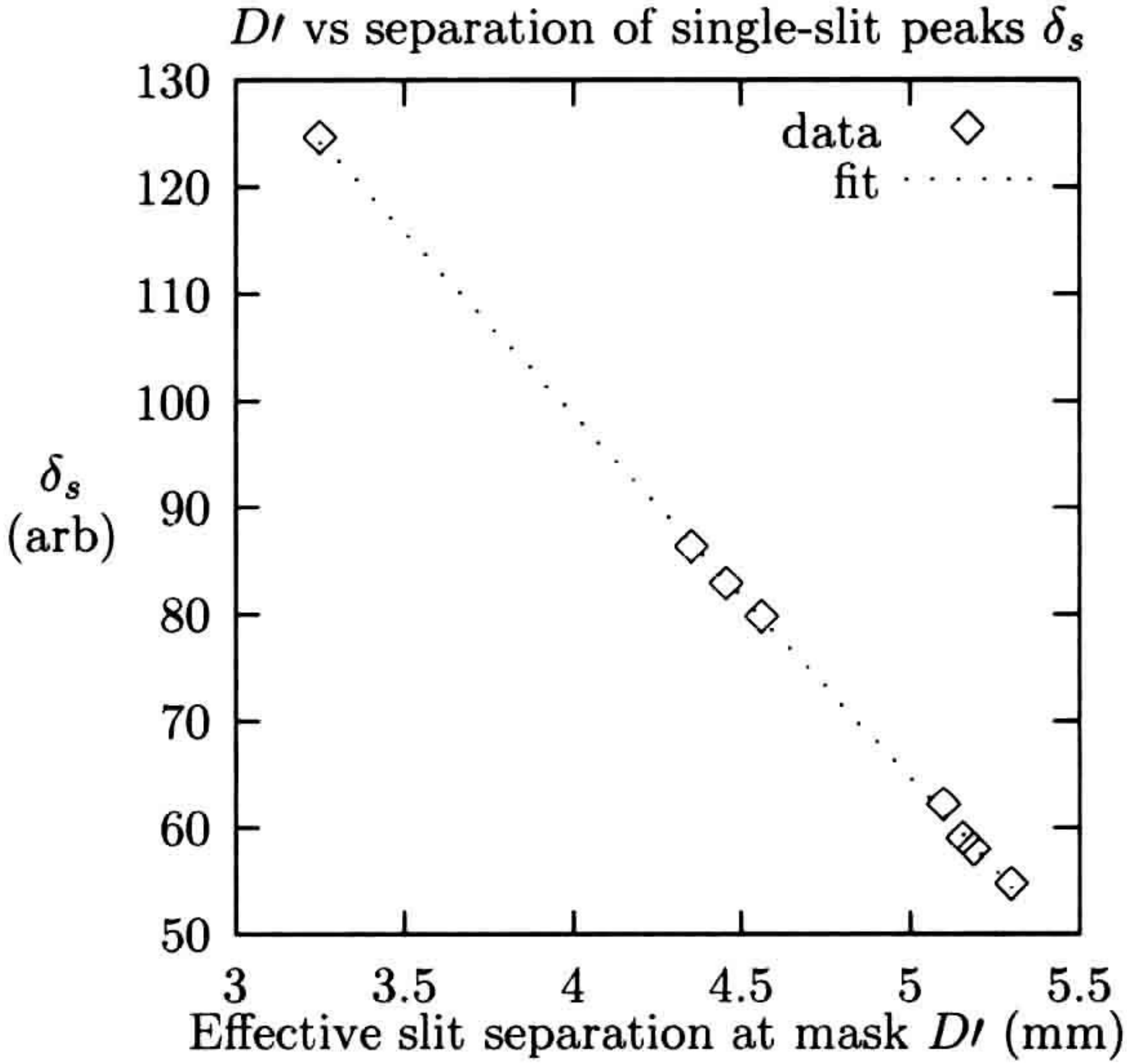
Normal observation plane

Defocusing observation plane


pcsrd04b
V. Intert Image
V. Interferometer
WiT Status
Error: gaussj: Singular Matrix-2
Error: srifit: mrqmin failed! (1)
y-axis
Data
Fit
x-axis
V. Intert Fit Parameters
float[26]
[0..4]: 290.425, -0.319412, -0.000647063, 45.0926, 0.151781,
[5..9]: 0.0151829, 0.00444358, 367.19, 187.944, 0.573103,
[10..14]: 0.637917, 344.155, 0.0996355, 0.000331185, 0.258423,
[15..19]: 0.00382906, 0, 0, 0.477319, 0.524902,
[20..24]: 0.00675551, 0.000138827, 0.0172205, 344, 125,
[25]: 35.0748
H. Intert Image
H. Interferometer
H. Intert Fit Parameters
float[26]
[0..4]: 205.597, -0.236783, -0.000794208, 49.076, 0.0548967,
[5..9]: 0.0169791, 0.0590663, 313.903, -106.397, 0.223677,
[10..14]: 0.400211, 331.325, 0.10522, 0.000244599, 0.192164,
[15..19]: 0.0101197, 0, 0, 0.745933, 0.623133,
[20..24]: 0.00367789, 0.000350492, 0.0414988, 331, 143,
[25]: 45.6305
zoomSize
throttle_caput
caputimage
caputvector
constant: BMHSRM:IMG:VINTERF
constant: BMHSRM:VEC:VINTER...
constant: BMHSRM:VEC:HINTER...
HSRM:IMG:HINTERF
type: sync
Standard: NTSC
Signal: Monochrom
Polarity: Negative
cumulative: Yes
showLegend: Yes
legendPos: (0, 0)
スタート
runwit
午後 12:14



KEKB Synchrotron Radiation Monitors
File
Edit
Window
10/04/2004 12:05:27
Help
HER Synchrotron Radiation Monitor
σy (µm)
σy @ IP (µm)
σx (µm)
σx @ IP (µm)
εy/εx (%)
Normal
V. Interferogram
V. Fit Parameters
V. Beam Size at IP
H. Interferogram
H. Fit Parameters
H. Beam Size at IP
Intensity (arb)
Pixel
Visibility
Correction
Current (mA)
LER Synchrotron Radiation Monitor
Menu Bar



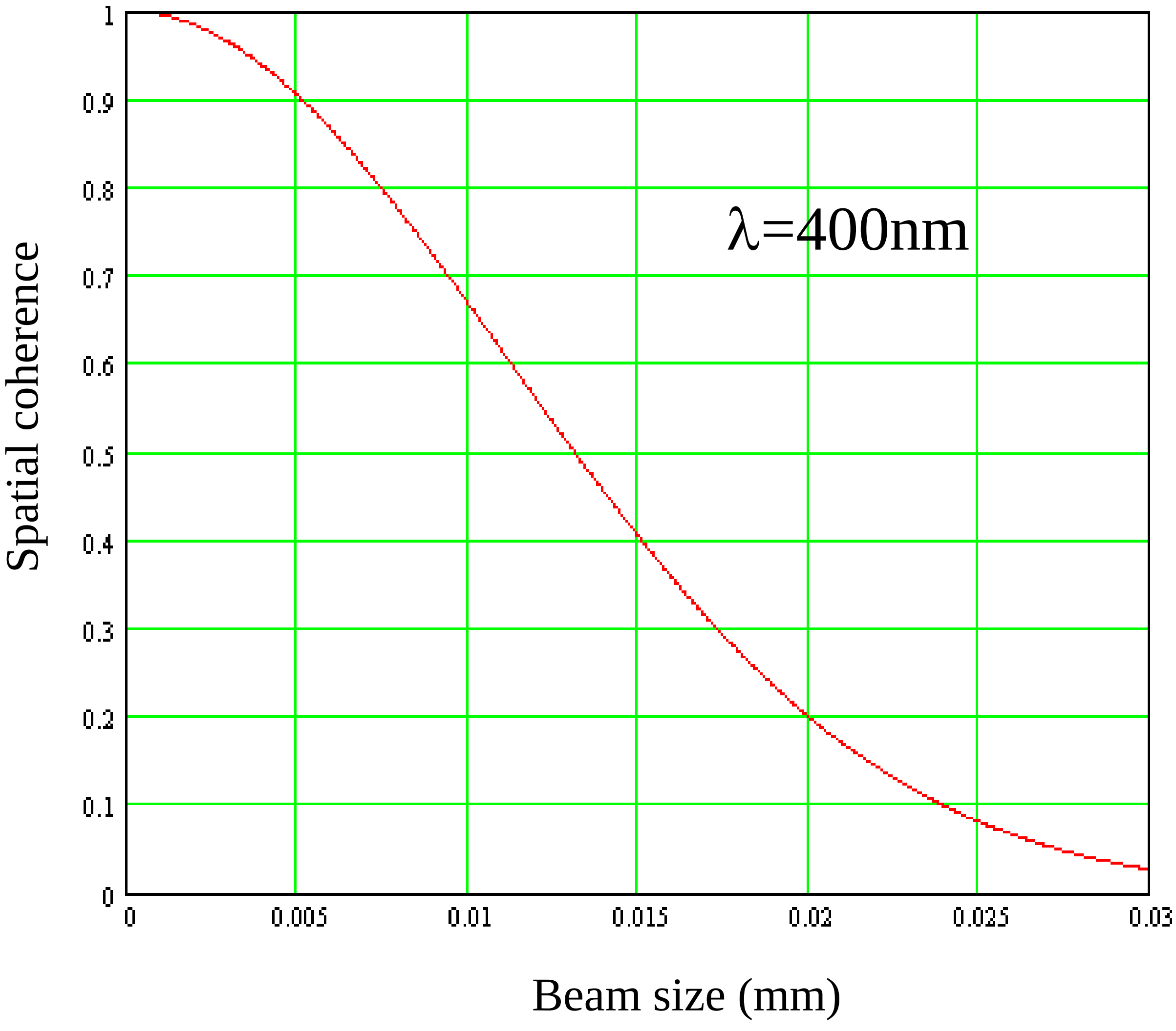
D′ vs separation of single-slit peaks δ_s
130
120
110
100
90
80
70
60
50
δ_s
(arb)
data
fit
3
3.5
4
4.5
5
5.5
Effective slit separation at mask D′ (mm)

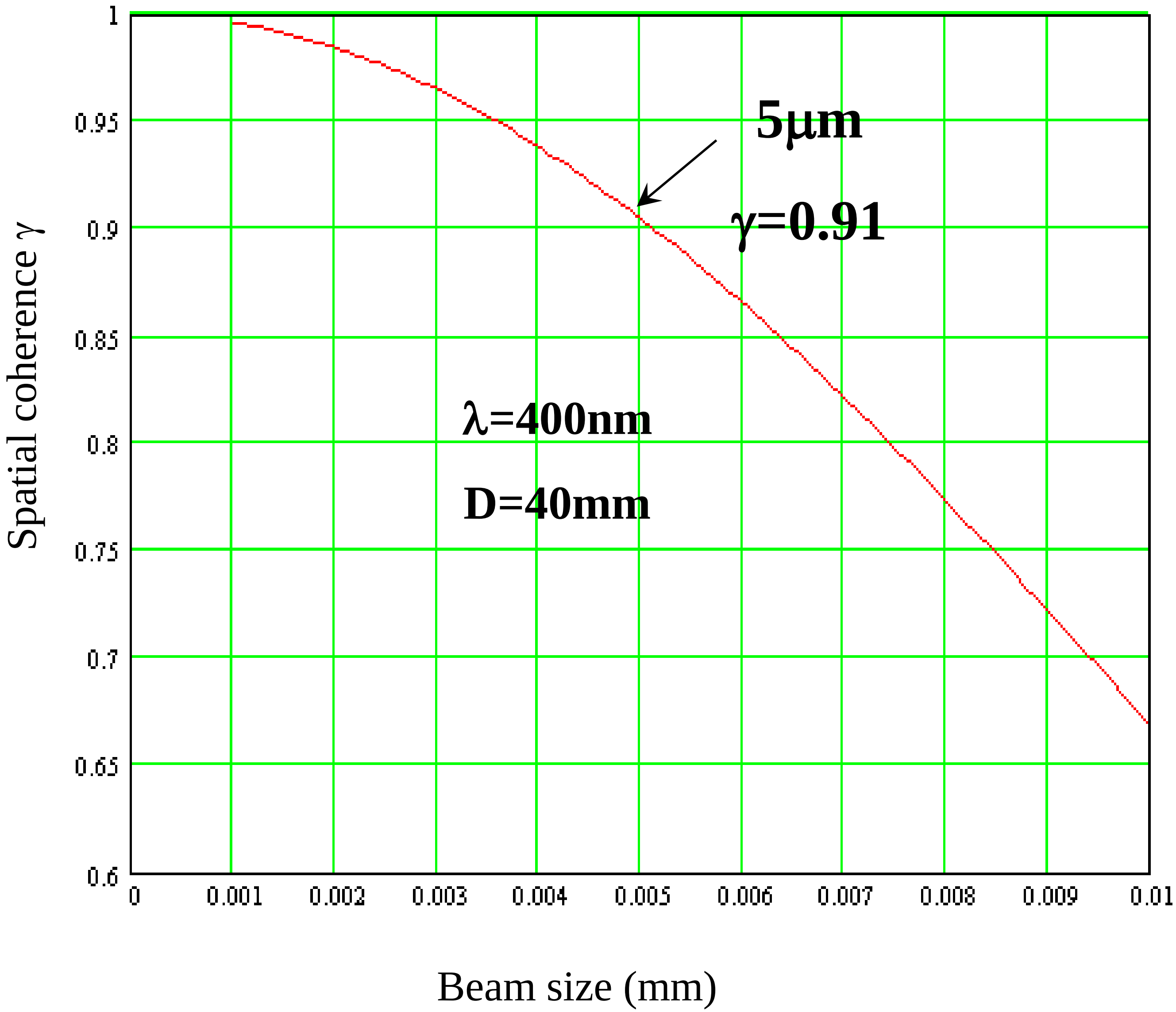
3-9. Other designs of optics for the SR interferometer

1. Reflective optics for eliminating the chromatic aberration for mesurement very small beam size at low ring current (few mA)
2. Retrofocus optics for the large beam size
3. Nonaplanatic optics

Very small beam size under week intensity of SR by using a reflective interferometer

Reflective optics to eliminate dispersion effect of glass under low intensity


λ=400nm
Spatial coherence
Beam size (mm)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.005
0.01
0.015
0.02
0.025
0.03



5μm
γ=0.91
λ=400nm
D=40mm
Spatial coherence γ
Beam size (mm)
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01

We often want to measure the beam size at small ring current (few mA).

In the case of the KEK-ATF, we use a band-pass filter of 80 nm to obtain sufficient intensity for the interferogram under the small ring current.

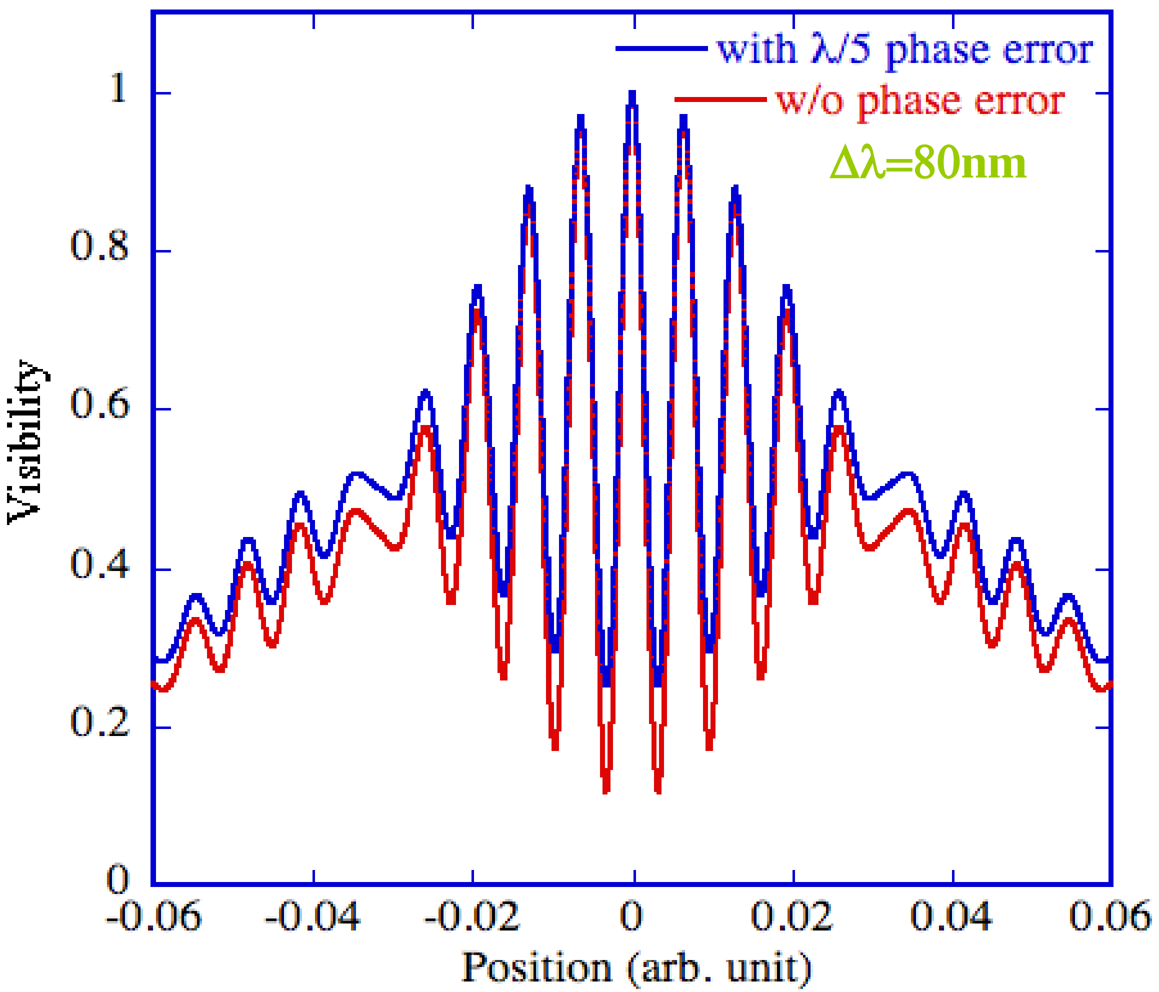
In measuring good visibility with wider window of band-pass filter, the most significant error arises from the chromatic aberration in the refractive optics.

Interferogram with chromatic aberration and without chromatic aberration.

λ=400nm, Δλ=80nm

Lens:achromat

D=80mm f=600mm



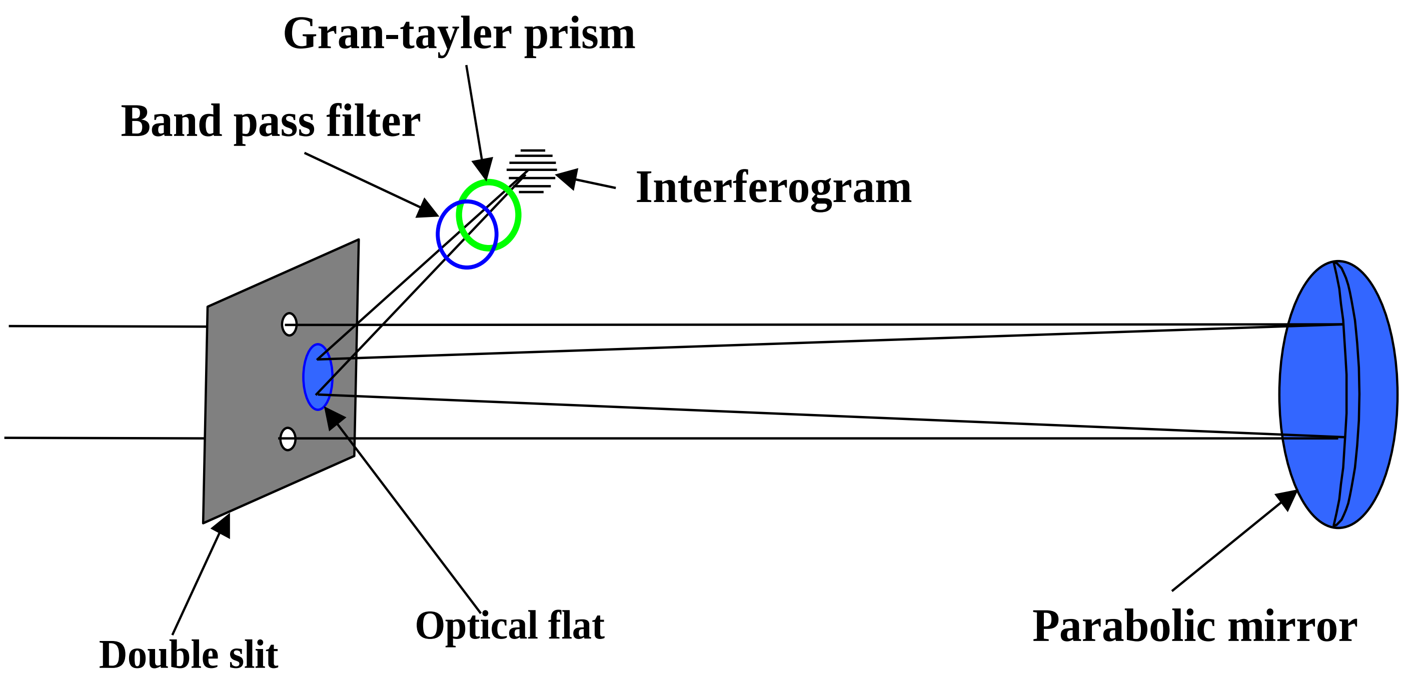
Elimination of chromatic aberration at 400nm is rather difficult due to large partial dispersion ratio of glass

↓

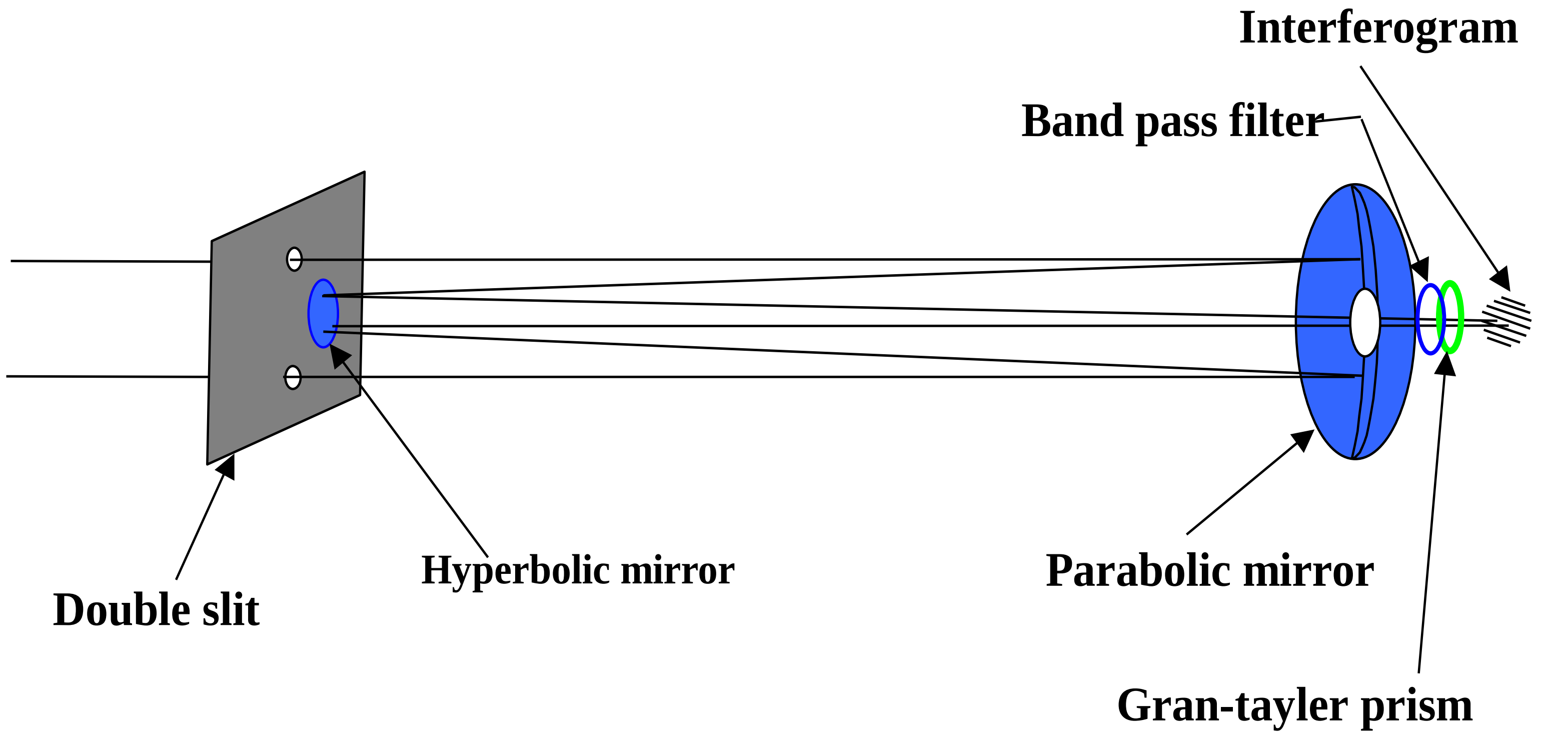
Use reflective optics

Some simple reflective optics for interferometer

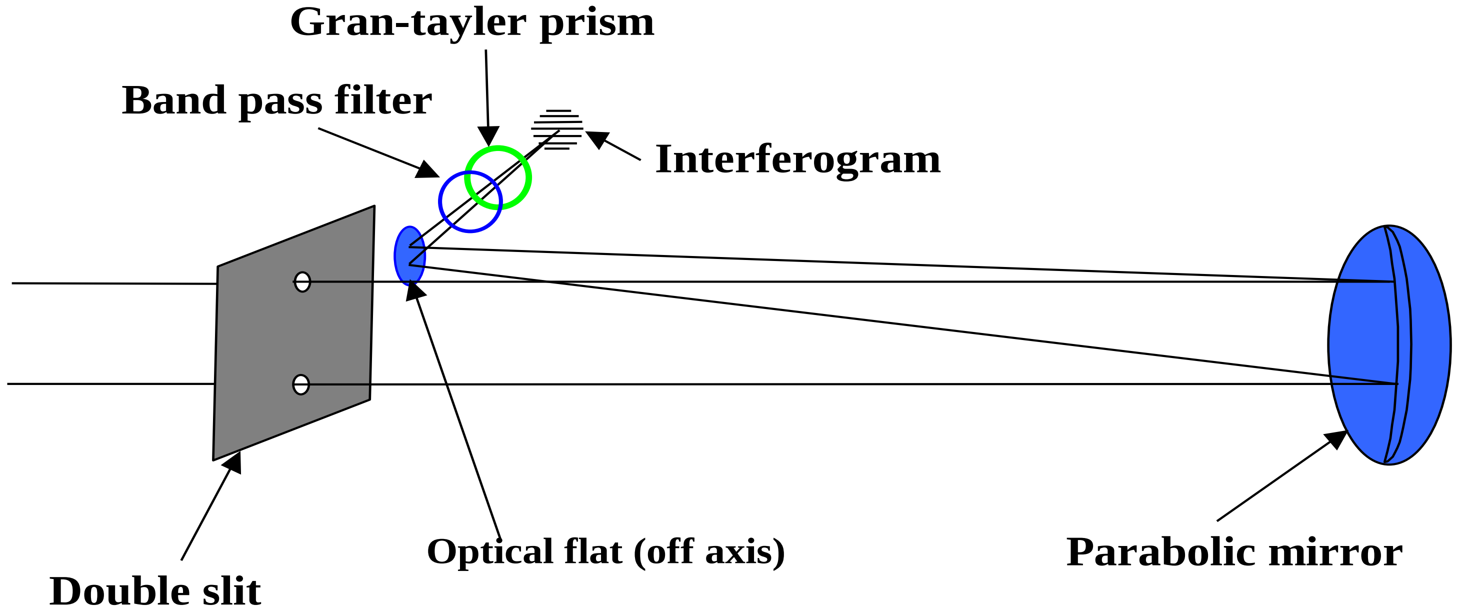
Newtonian arrengement of optics



Cassegrainian arrangement of optics



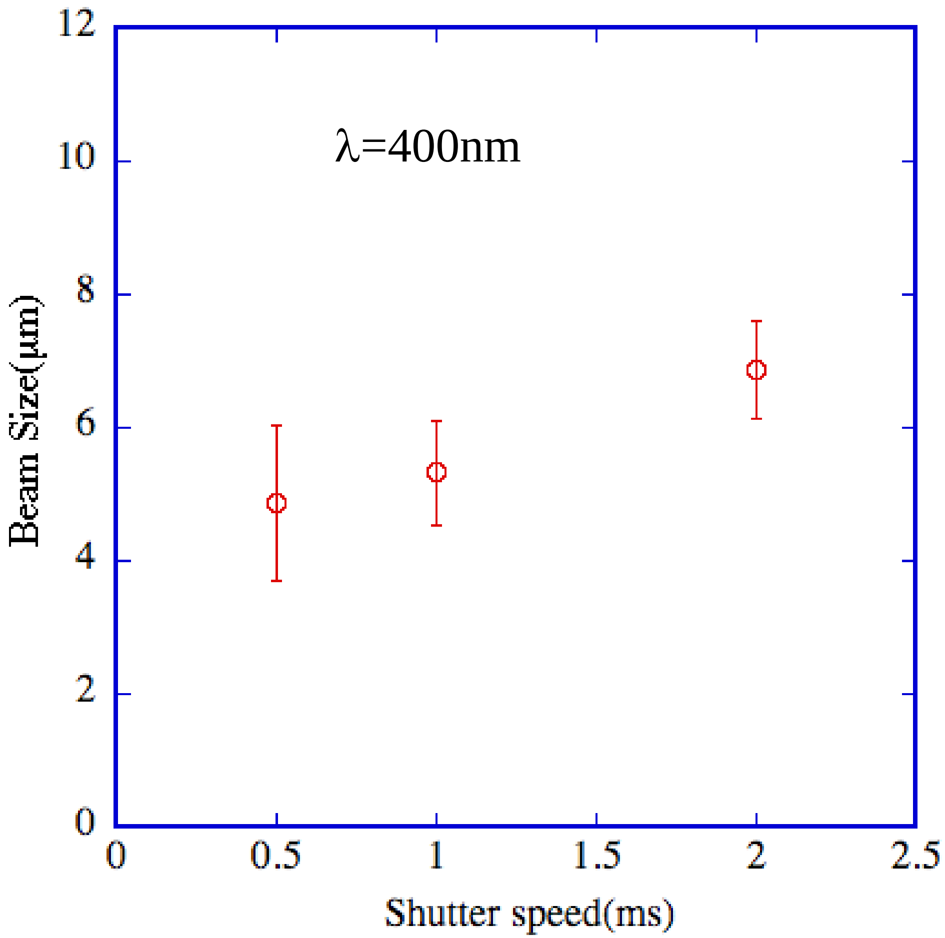
Herschelian arrengement of optics



Results of small beam size measurement by Hershelian interferometer

Minimum measured beam size is 4.73μm±0.55μm

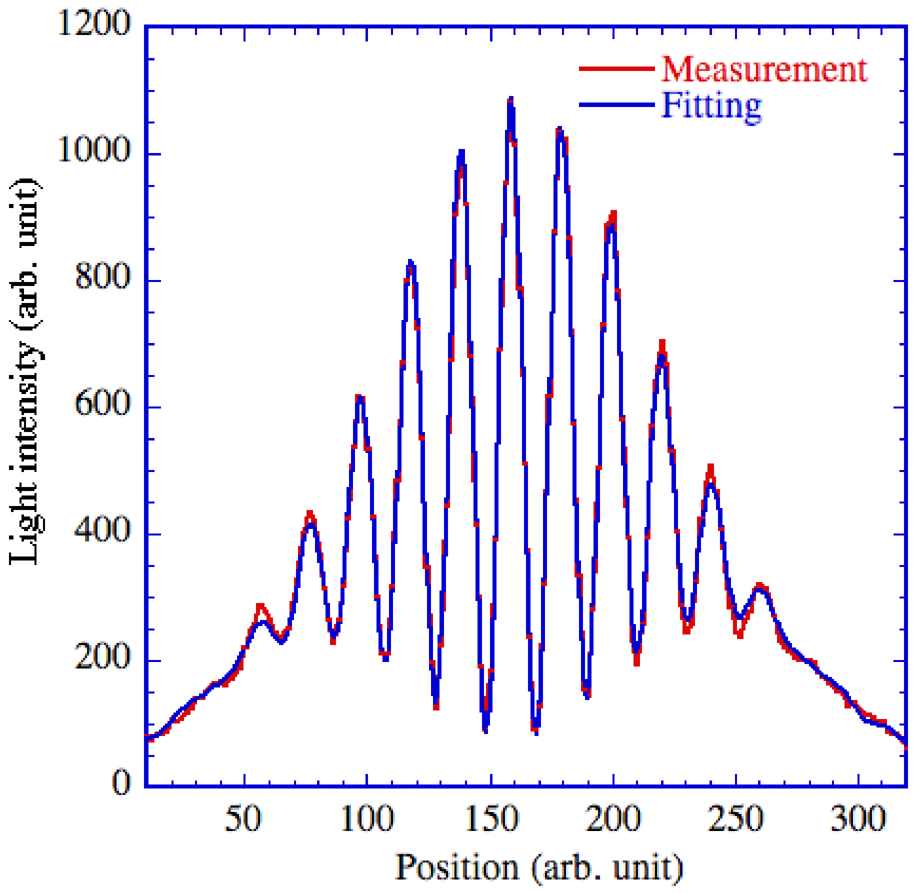
Corresponding emittance is 9.7pmrad



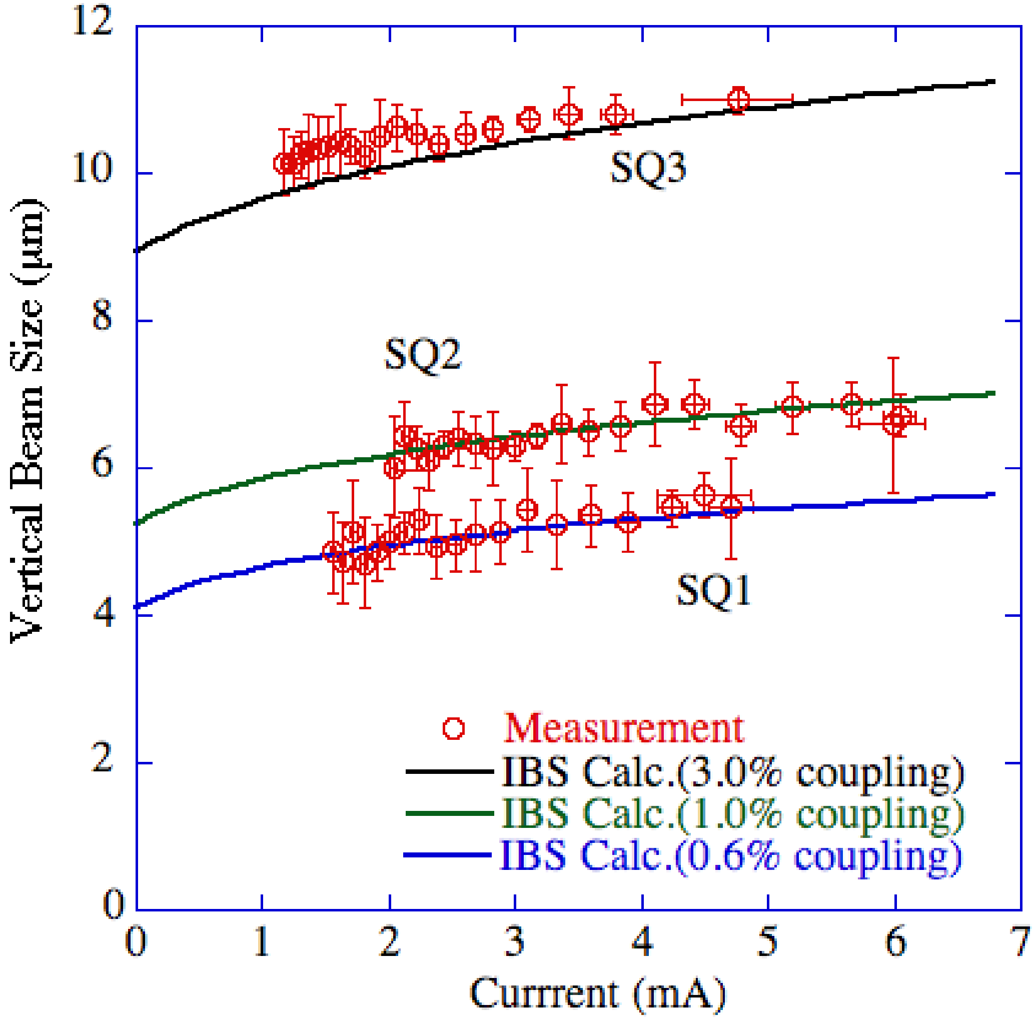
Measured interferogram

Result of beam size is 4.73μm±0.55μm

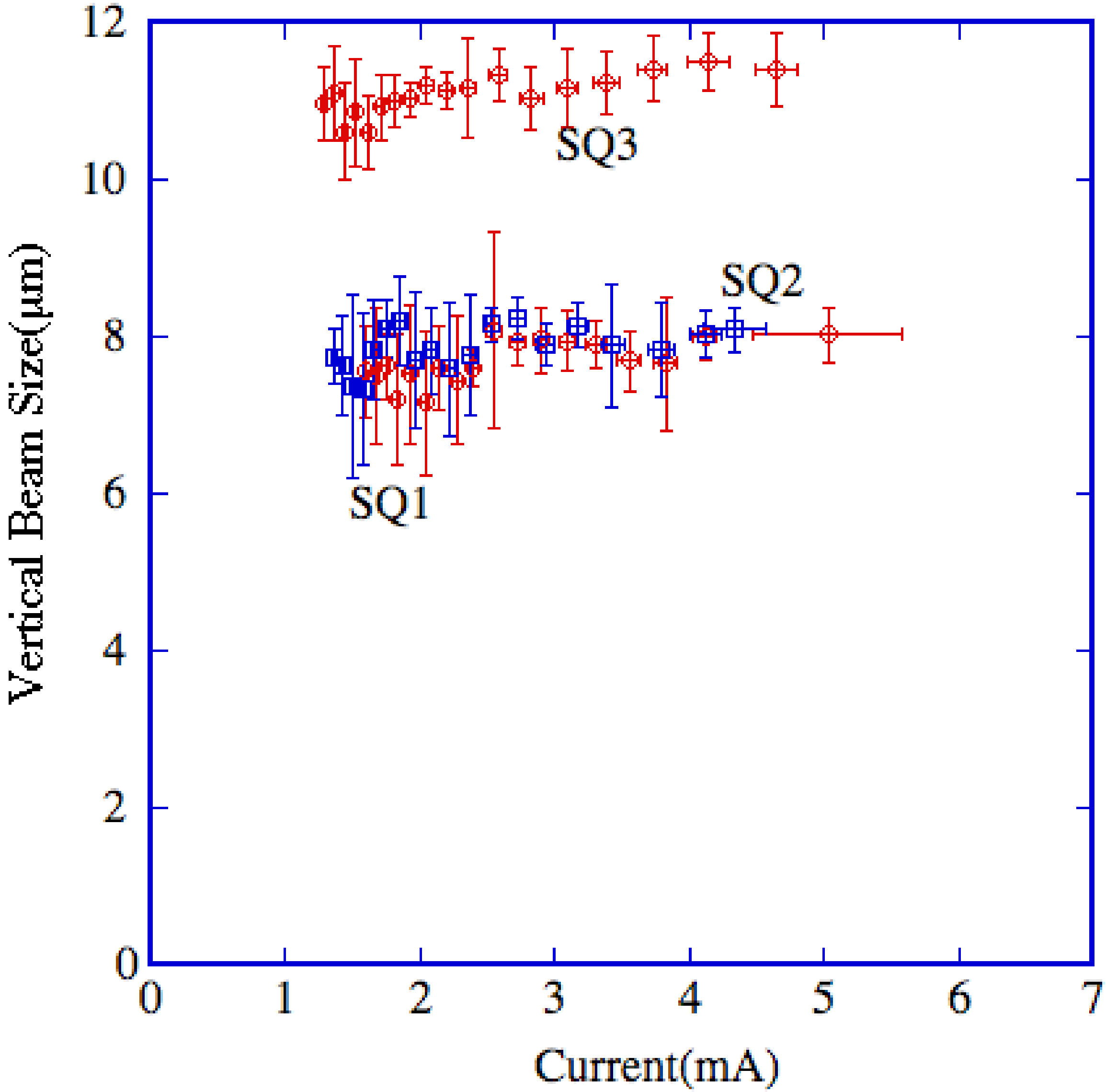
Corresponding emittance is 9.7pmrad



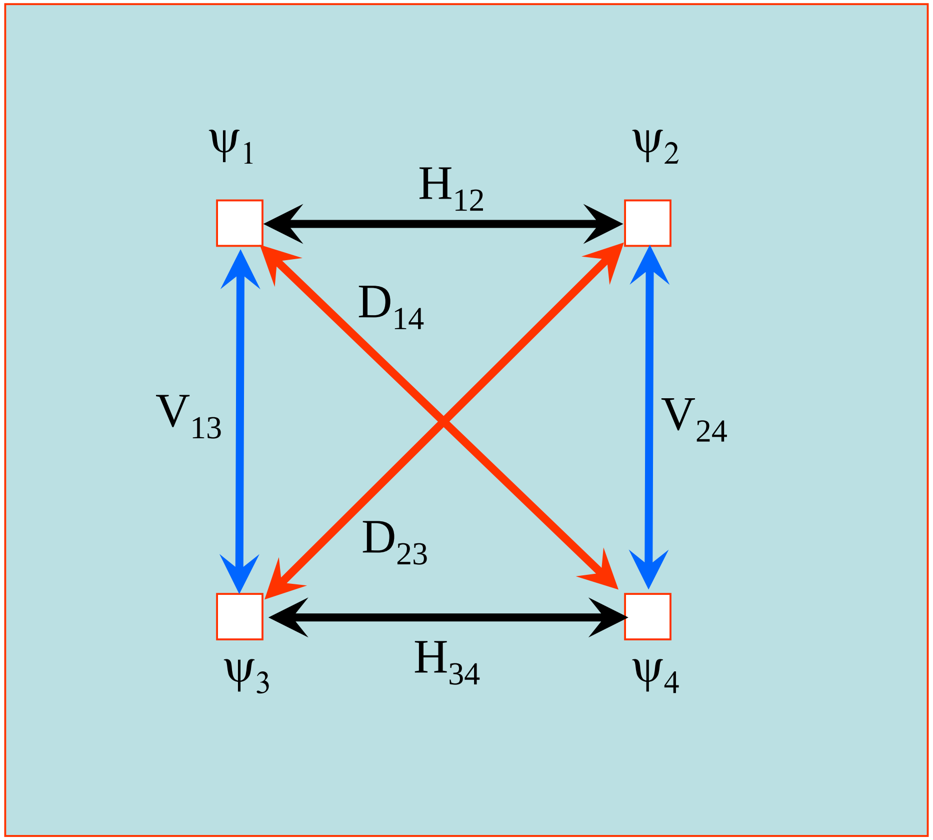
The x-y coupling is controlled by the strength of the skew component of sextupole magnets(SD, SF) at ATF



Same results by normal refractive interferometer using λ=400nm



3-10. Two dimensional interferometer



$$I = \psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2$$
$$+ H_{12} + H_{34} + V_{13} + V_{24} + D_{14} + D_{23}$$

In here,

$\psi_2=\psi_1^*$

$\psi_3=\psi_1^*$

$\psi_4=\psi_3^*=\psi_1$

$H12= |\psi_1 \cdot \psi_1^*|_{D1}$

$H34= |\psi_1^* \cdot \psi_1|_{D1}$

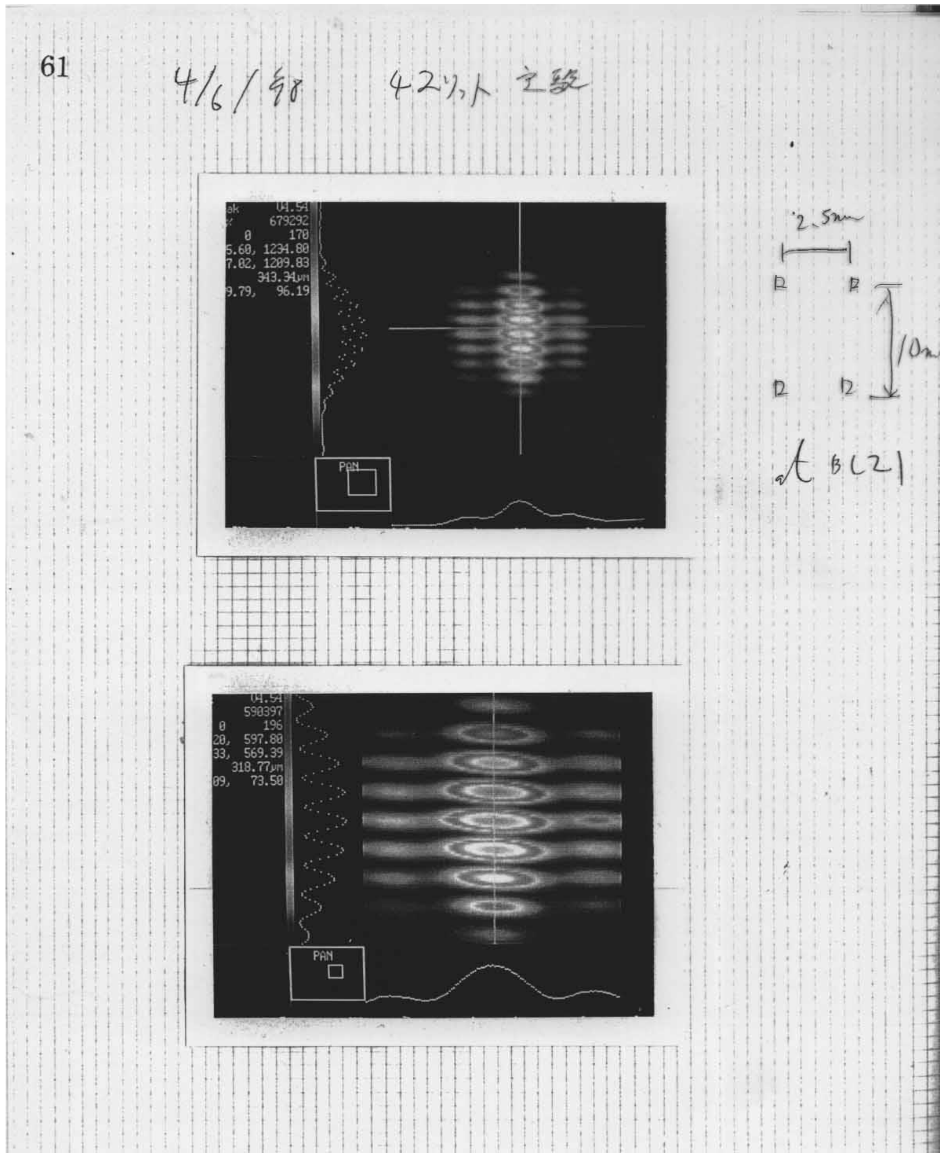
$V13=|\psi_1 \cdot \psi_1^*|_{D2}$

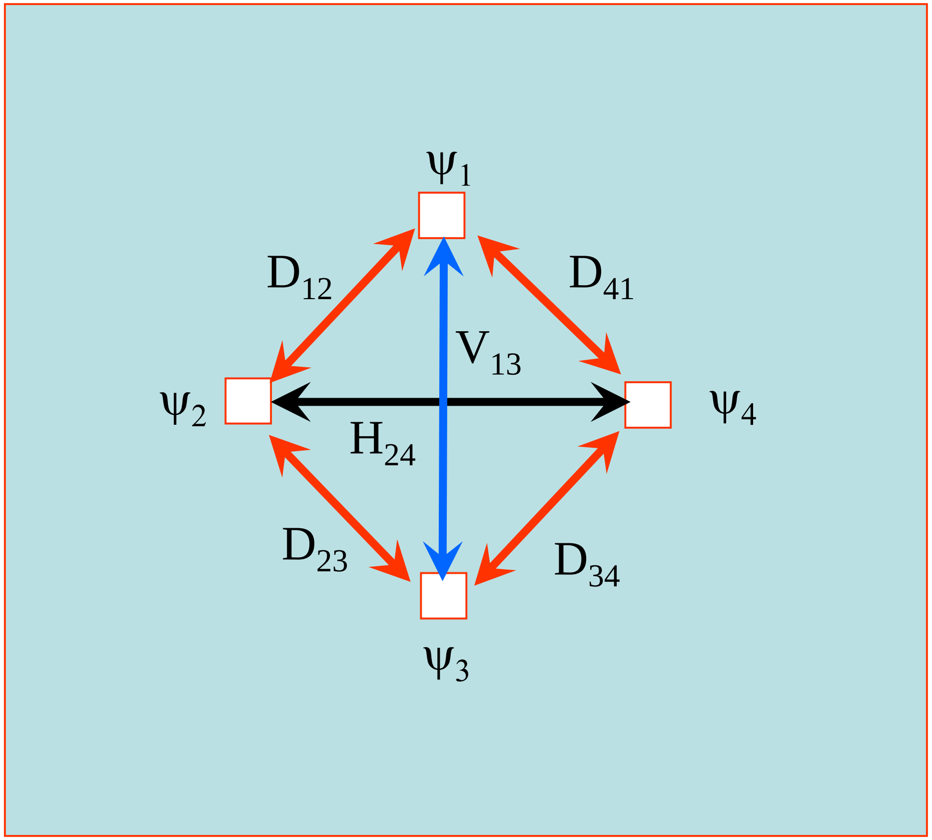
$V24= |\psi_1^* \cdot \psi_1|_{D2}$

$D14= \psi_1 \cdot \psi_1|_{D3}$ unobservable when field has no aberration

$D23= \psi_1^* \cdot \psi_1^*|_{D3}$ unobservable when field has no aberration

2-D interferometer experiment April,1998





Can we choose this configuration?

In here,

$\psi_3 = \psi_1^*$

$\psi_4 = \psi_2^*$

?????

In this configuration,

If,

$\psi_3 = \psi_1^*$

$\psi_4 = \psi_2^*$

$\psi_2 = \psi_1$

$\psi_4 = \psi_1^2$

$H_{12} = |\psi_1 \cdot \psi_1^*|_{D1}$

$H_{34} = |\psi_1^* \cdot \psi_1|_{D1}$

$V_{13} = |\psi_1 \cdot \psi_1^*|_{D2}$

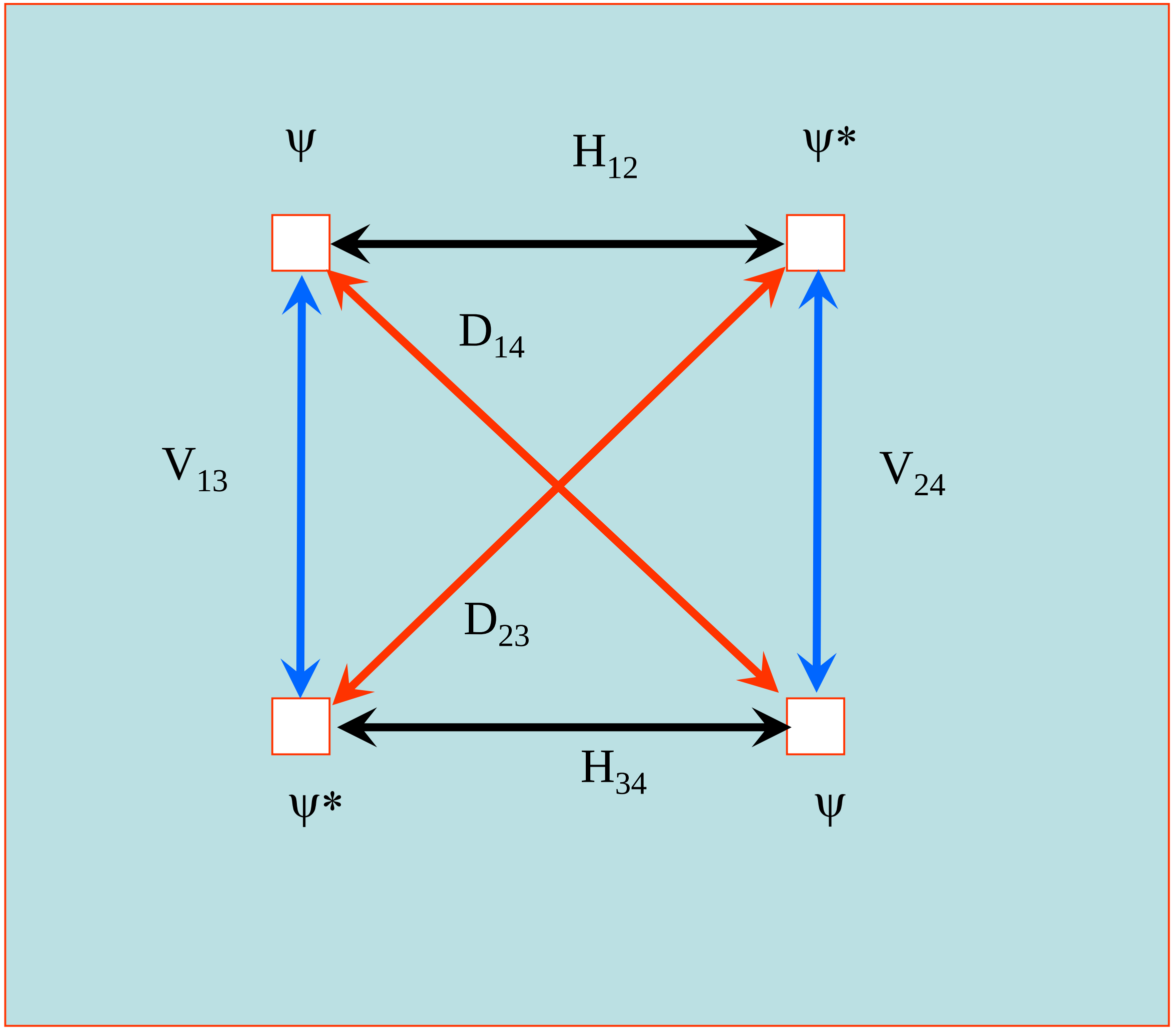
$V_{24} = |\psi_1^* \cdot \psi_1|_{D2}$

$D_{14} = \psi_1 \cdot \psi_1|_{D3}$ unobservable when field has no aberration

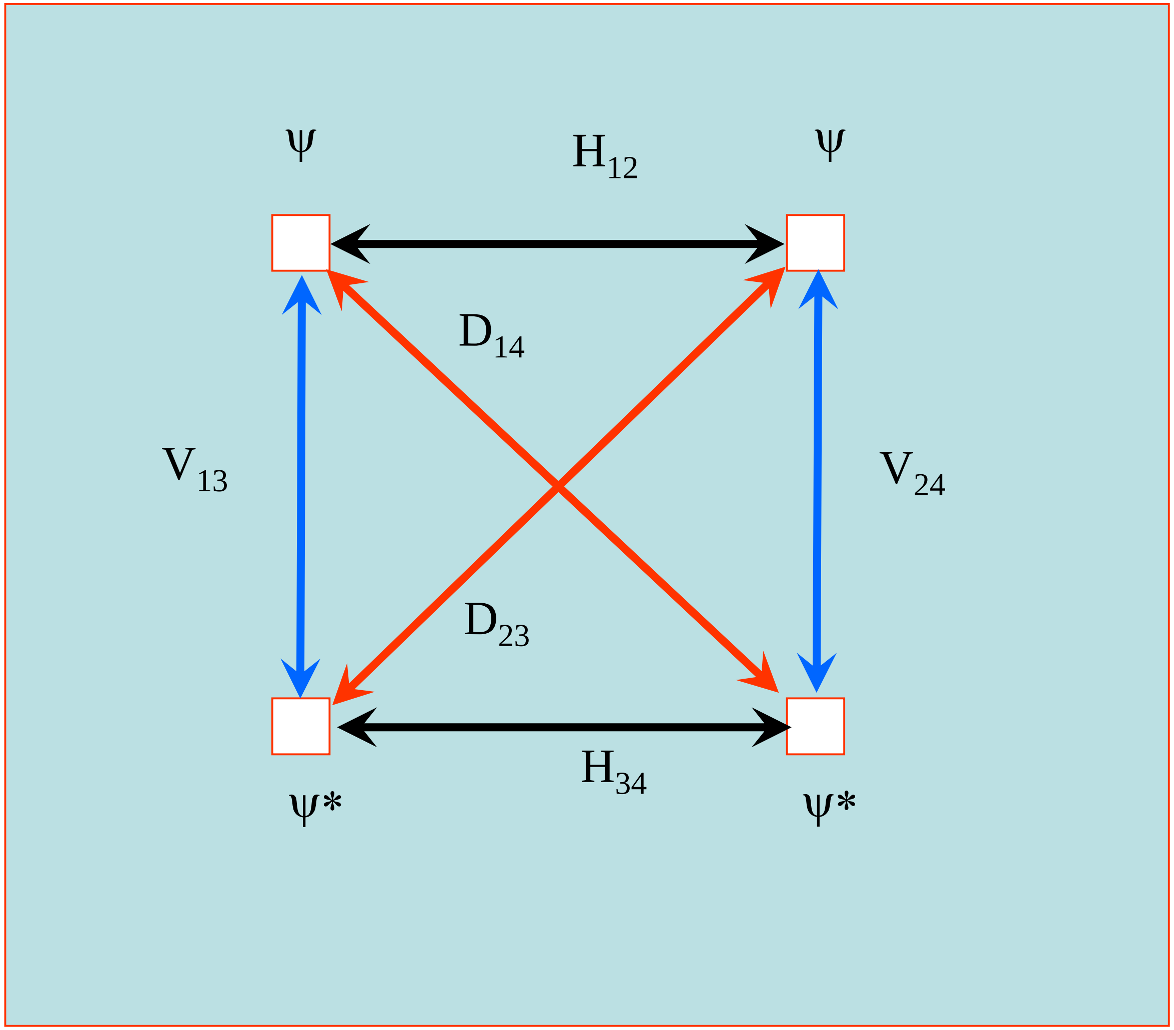
$D_{23} = \psi_1^* \cdot \psi_1^*|_{D3}$ unobservable when field has no aberration

Is it correct?? ➡ No!!

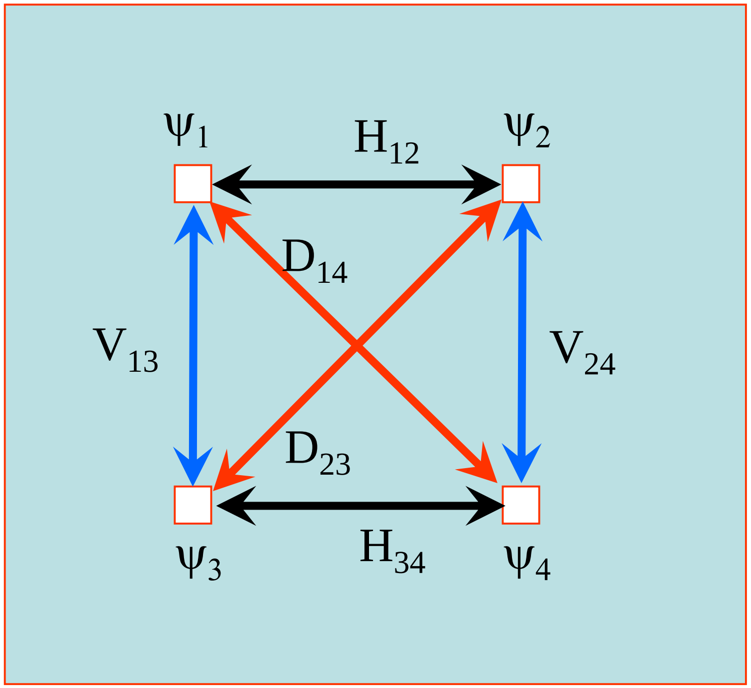
We must choose 4 fog modes of one photo as follows;



Not like this way;



We cannot choose complex conjugate pair in the diagonal product!!



In other wise,

Using this scheme, If, wavefront error will existing (actually, it will be happen very often) between 1-2 and 2-4, or (and) 1-3 and 2-4, H_{12}will not equal to H_{34}, or (and) V_{13} will not equal to V_{24}.

Also, D_{14} and D_{14} will have real component (Observable).

In this situation, we cannot measure beam size correctly.

Merit and demerit in 2D interferometer

Merit

1) We can measure vertical and horizontal beam sizes simultaneously.

Demerit

1) We cannot change the conditions of interferometer such as slit separation independently.

2) It seems very difficult to perform the slit scanning.

3) If vertical beam size and horizontal beam size are something different, other configuration of the interferometer will necessary(for example retro-focus type interferometer).

Conclusion

It seems convenient to use two independent interferometers!

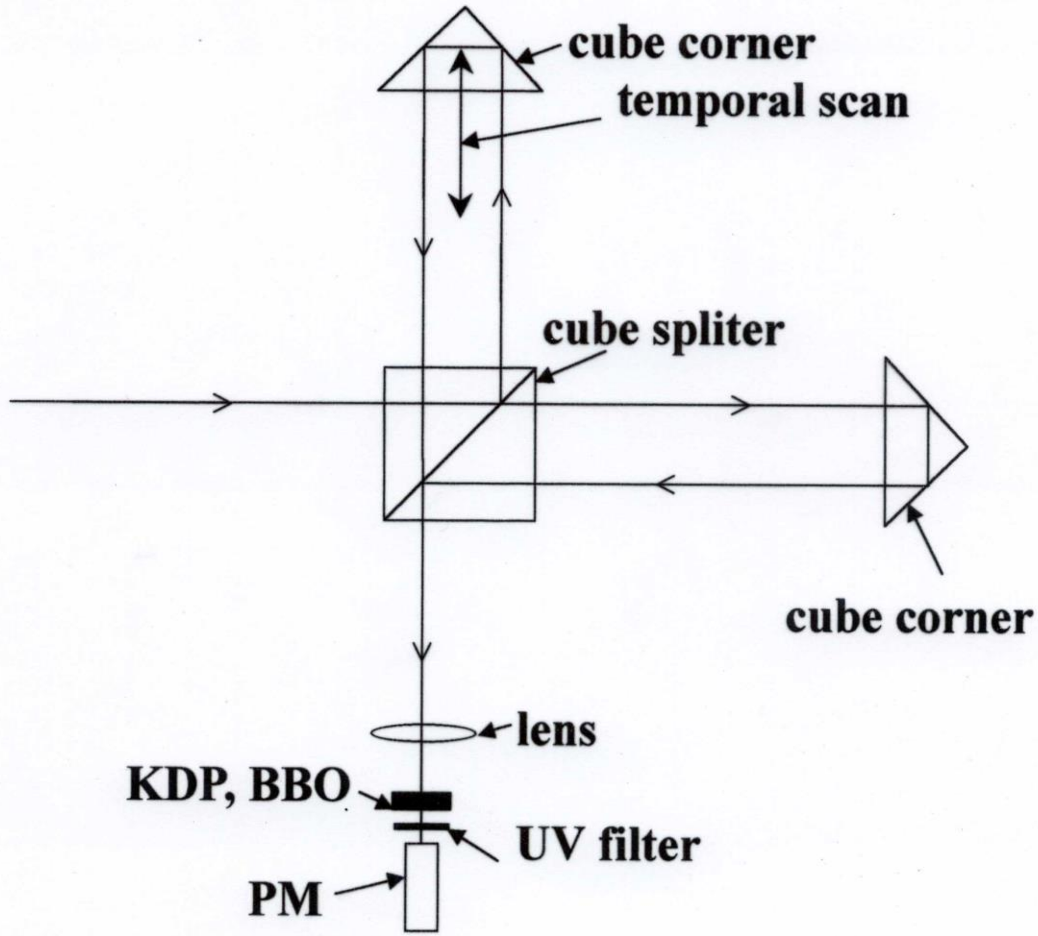
Intensity interferometry to measure short bunch length

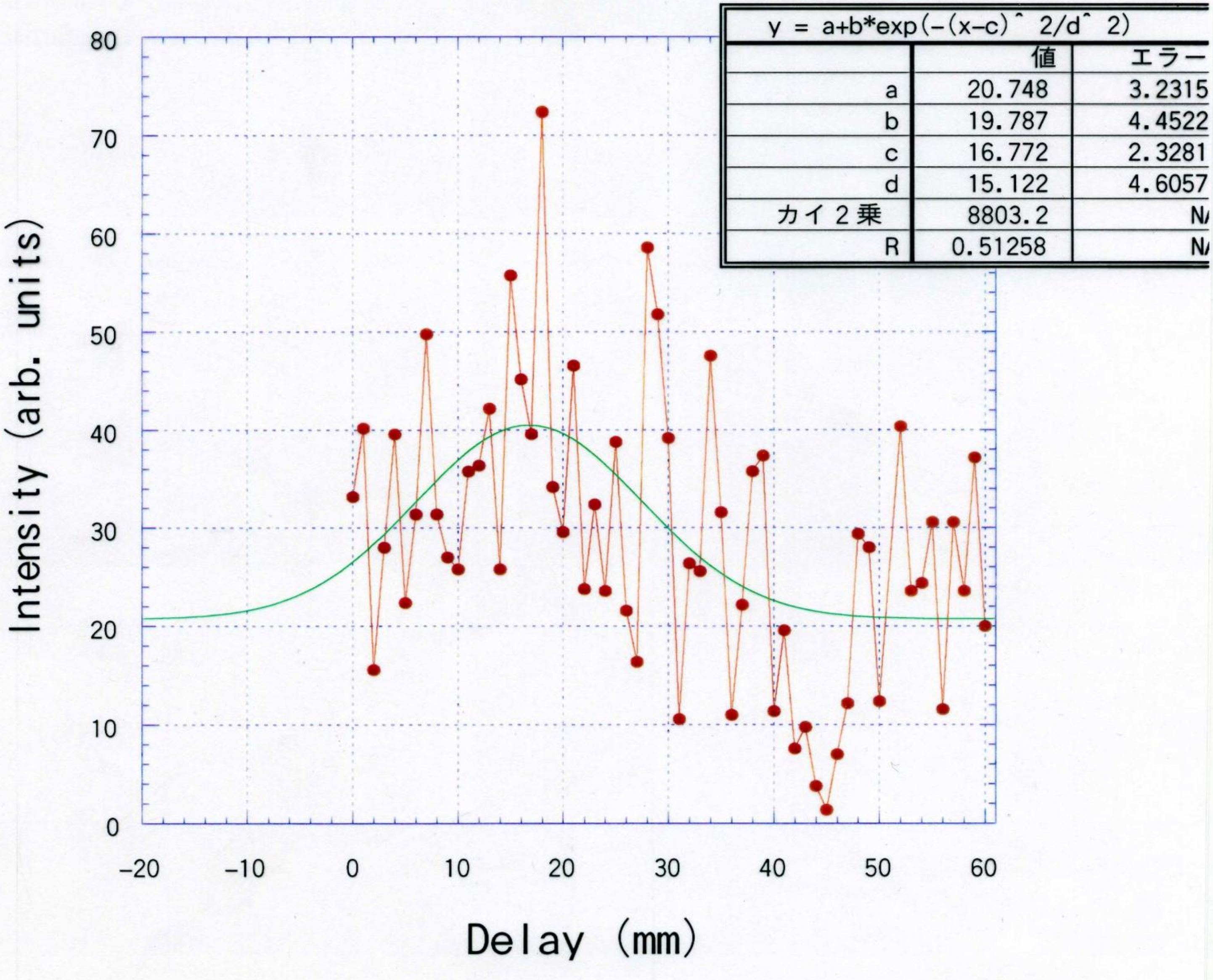
Intensity interferometry experiment

1. Investigate the photon statistics in the synchrotron radiation.

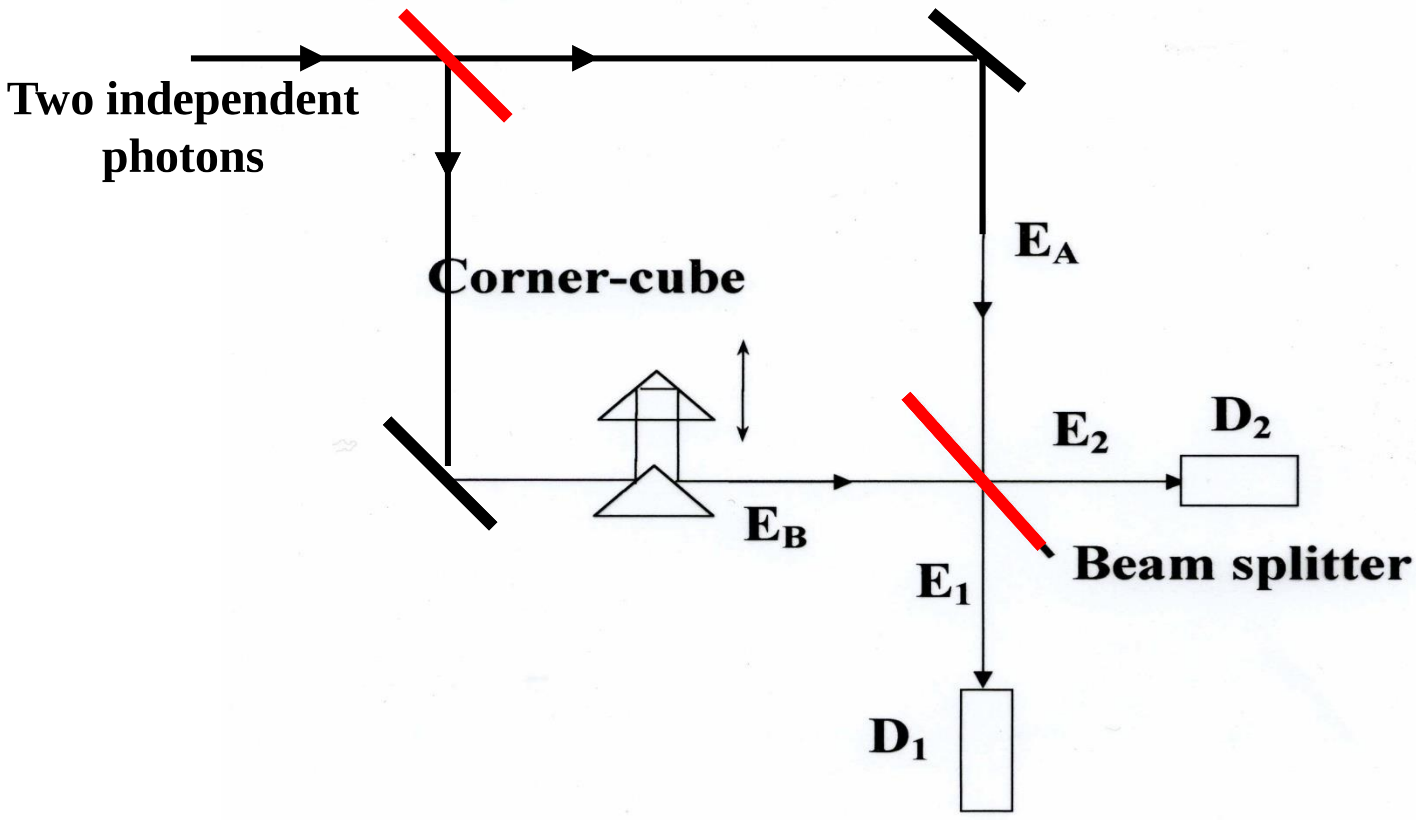
2. Demonstrate possibility to measure very short bunch length (fs region).

Principle of second-order autocorrelator: colinear arrangement




y = a+b*exp(-(x-c)^ 2/d^ 2)
値
エラー
a 20.748 3.2315
b 19.787 4.4522
c 16.772 2.3281
d 15.122 4.6057
カイ２乗 8803.2 N/
R 0.51258 N/
80
70
60
50
40
30
20
10
0
-20
-10
0
10
20
30
40
50
60
Intensity (arb. units)
Delay (mm)

Bunch length measurement by intensity interferometry

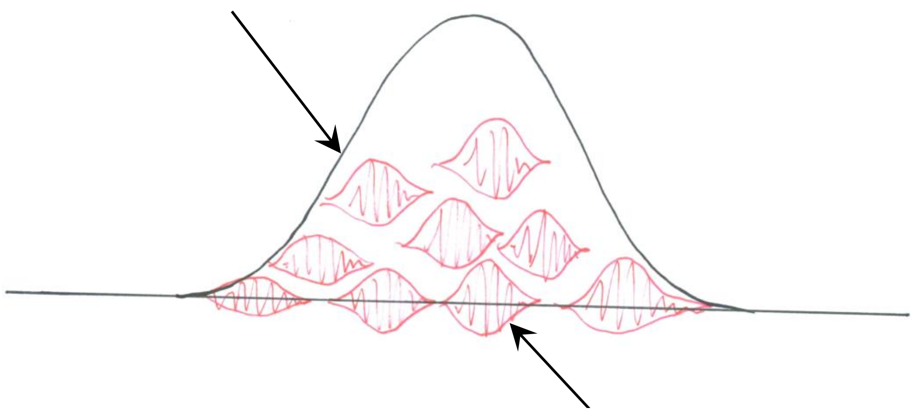


Fields of input lights E_A, E_B

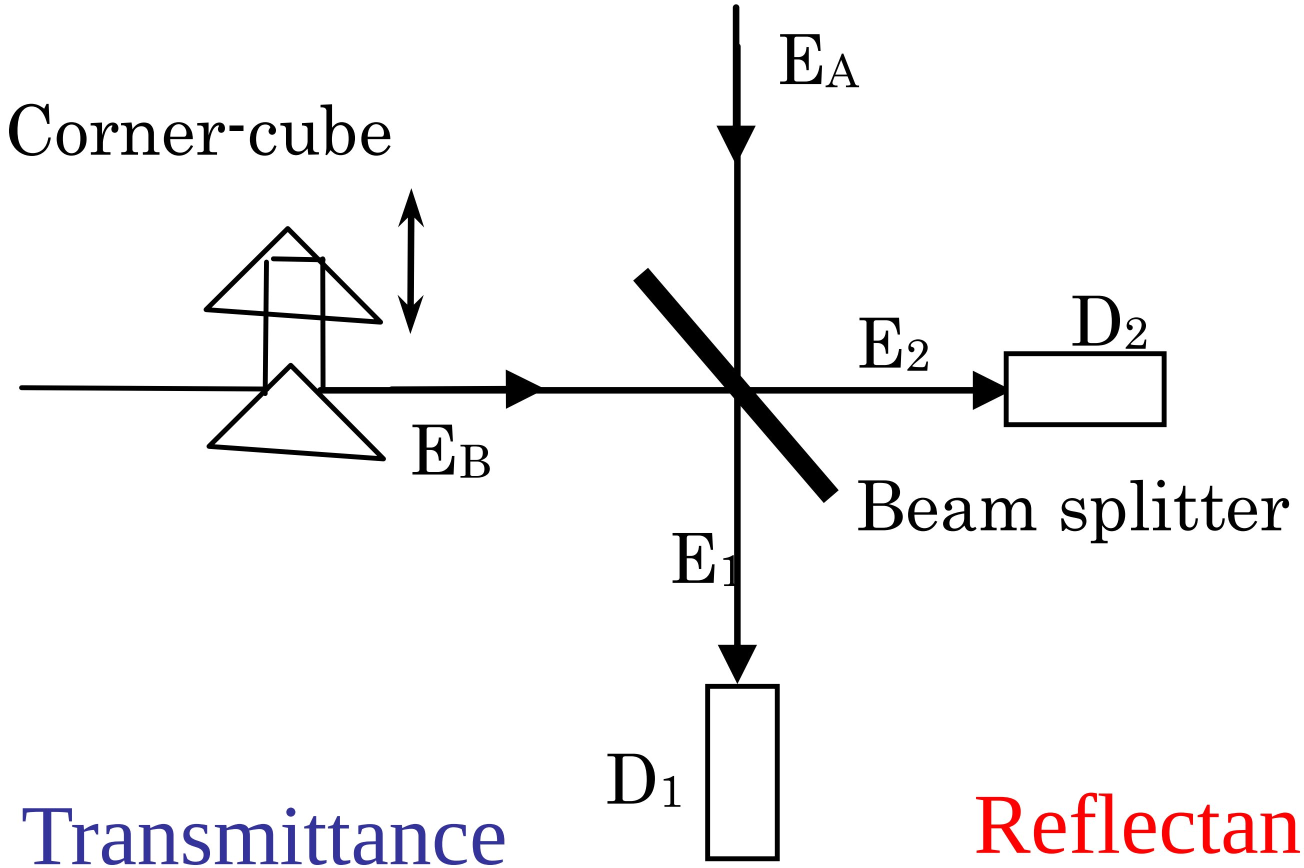
$$E_A(t) = C_A(t)A_A(t)$$

$$E_B(t) = C_B(t)A_B(t) \; .$$

Pulse envelope C(t) having a pulse duration of σ_p



stationary random variable A(t) having a coherent length of τ_c



Transmittance

Reflectance

$$E_1(t) = \sqrt{T} \cdot E_A(t) + i\sqrt{R} \cdot E_B(t + \delta\tau)$$
$$E_2(t) = \sqrt{T} \cdot E_B(t + \delta\tau) + i\sqrt{R} \cdot E_A(t) \ .$$

Using E1and E2, the simultaneous count Count12(dt) of detector D1 and detector D2, is given by;

$$Count_{12}(\delta\tau) = K\int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} dt \int_{-\frac{T_r}{2}}^{\frac{T_r}{2}} d\tau \,\langle E_1{}^{*}(t) E_2{}^{*}(t+\tau) \times E_2(t+\tau) E_1(t)\rangle \;,$$

Let assume both of C(t),A(t) be the Gaussian , and the input light fields E_A,E_B have the first order temporal coherence, then perform the integration;

Correlation term between the pulse duration

$$count_{12}(\delta\tau) = K\sigma_p{}^2 \times$$

$$\left[1 - \boxed{\frac{1}{2}\exp\left(-\frac{\delta\tau^2}{4\tau_c{}^2}\right)} + \frac{\tau^*}{\sigma_p}\left(1 - \boxed{\frac{1}{2}\exp\left(-\frac{\delta\tau^2}{4\sigma_p{}^2}\right)}\right)\right]$$

Correlation term between the wavepockets

$$\frac{1}{\tau^{*2}} = \frac{1}{\sigma_p{}^2} + \frac{1}{\tau_c{}^2}.$$

When both of E_A,E_B has no first order temporal coherence, the correlation term between the wavepocket will disappear,

then the simultaneous count is given by;

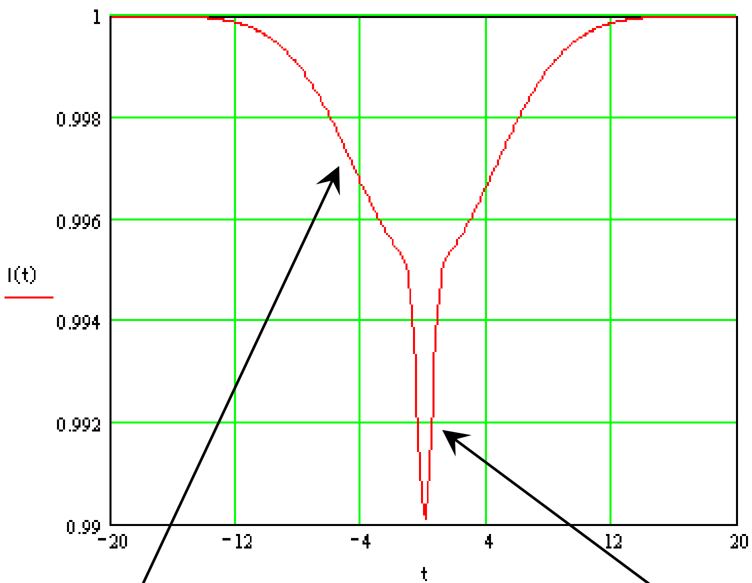
$$Count_{12}(\delta\tau) = K\sigma_p^{\,2}\left(1+\frac{\tau^{*}}{\sigma_p}\left[1-\boxed{\frac{1}{2}\exp\left(-\frac{\delta\tau^{2}}{4\sigma_p^{\,2}}\right)}\right]\right),$$

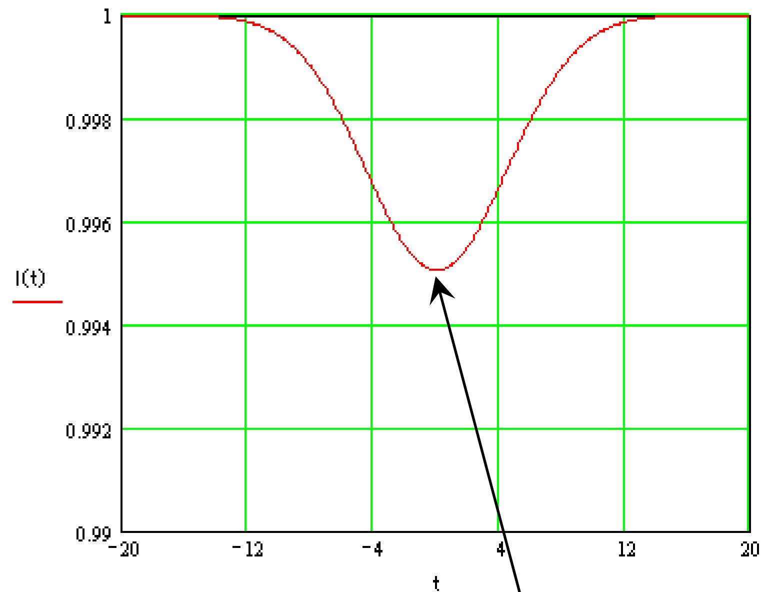
Correlation term between

the pulse duration

E_A,E_B has first order temporal coherence

E_A,E_B has no first order temporal coherence



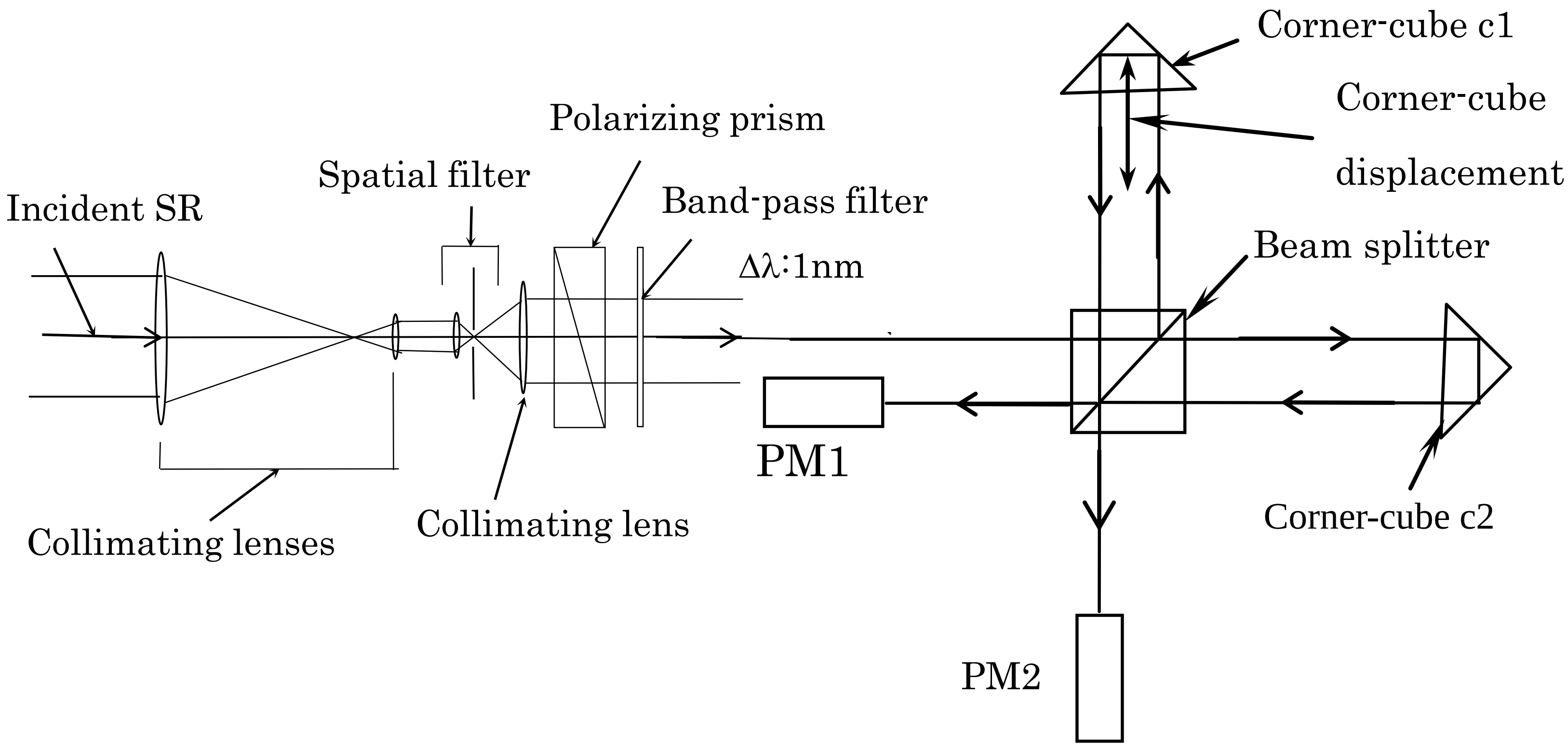


Auto-correlation between the pulse duration

Auto-correlation between the wavepockets

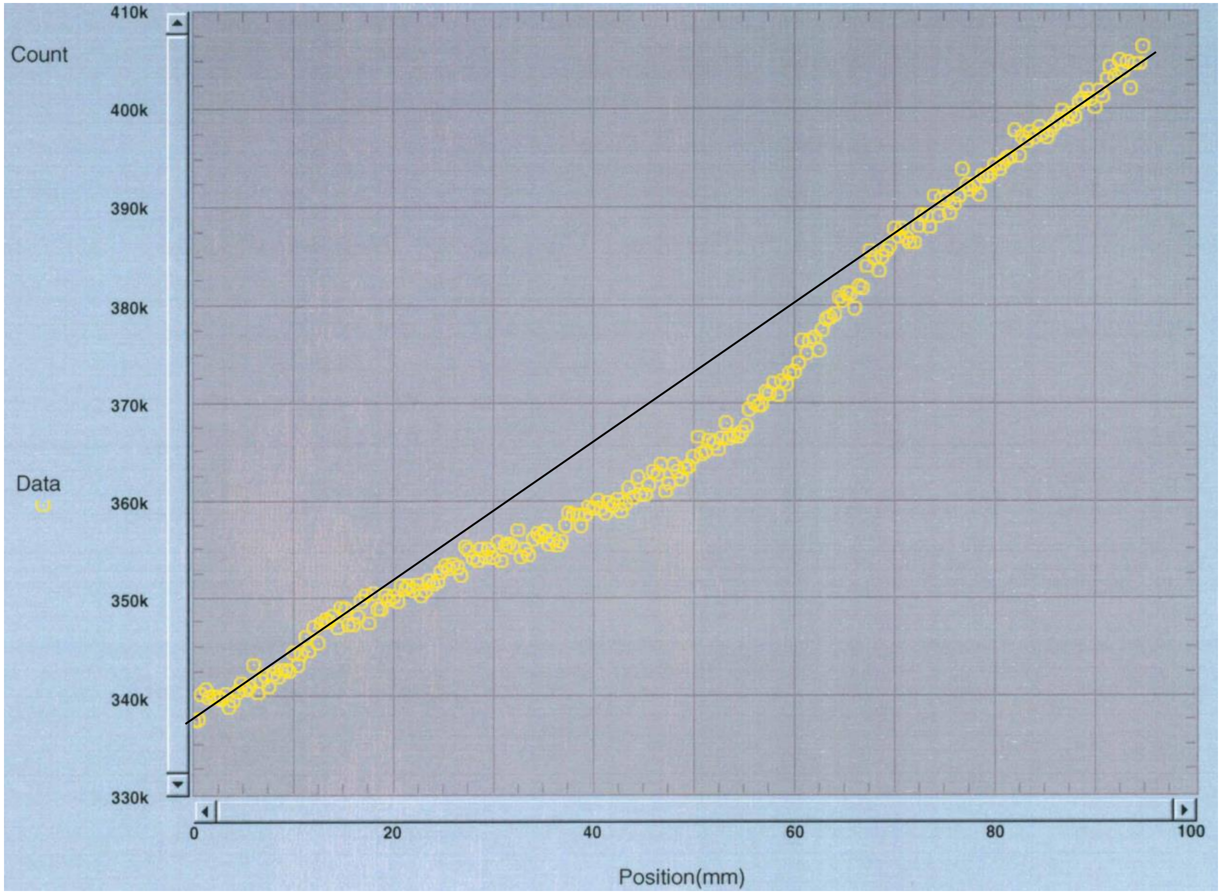
Only the auto-correlation between the pulse duration will remain

Actual setup of the intensity interferometer



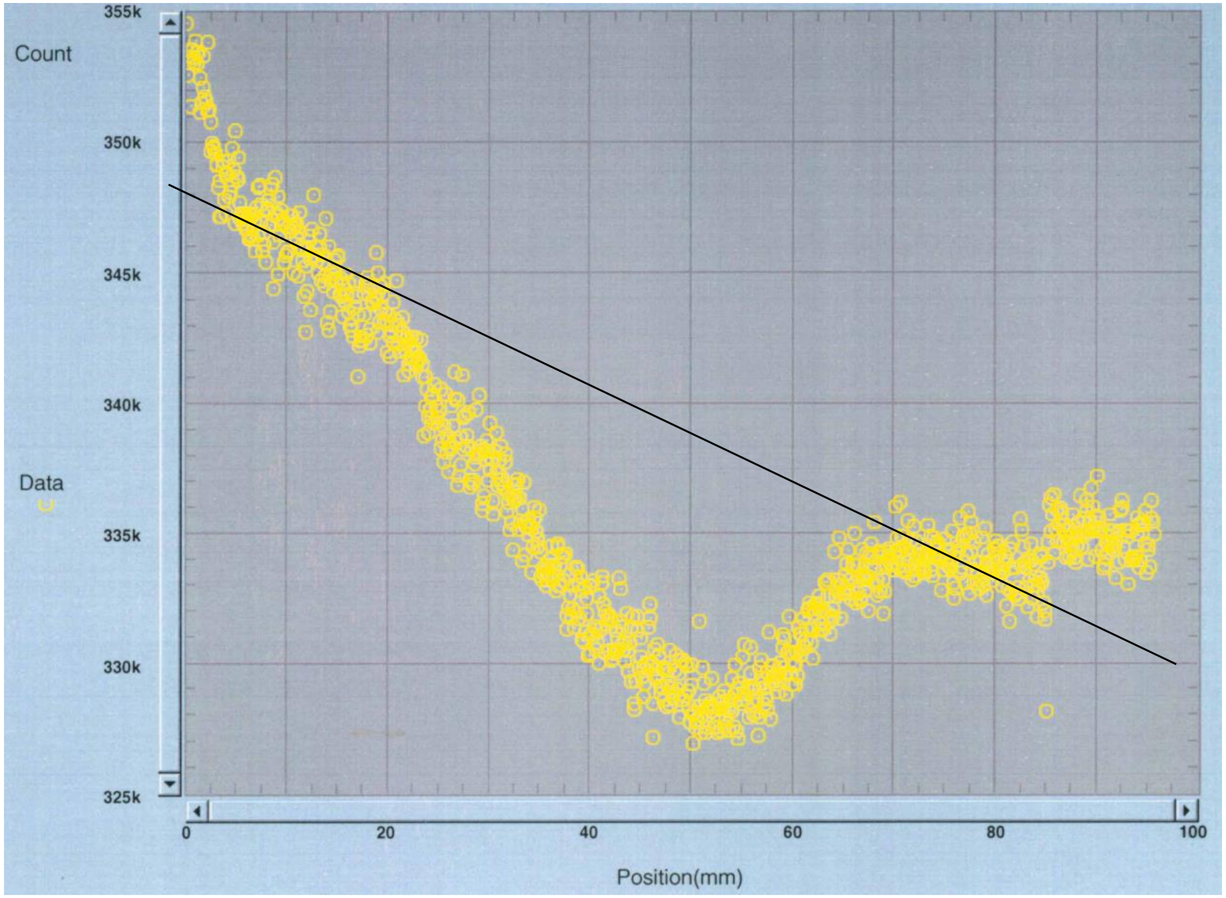
Example of Count$_{12}$

Over collimation

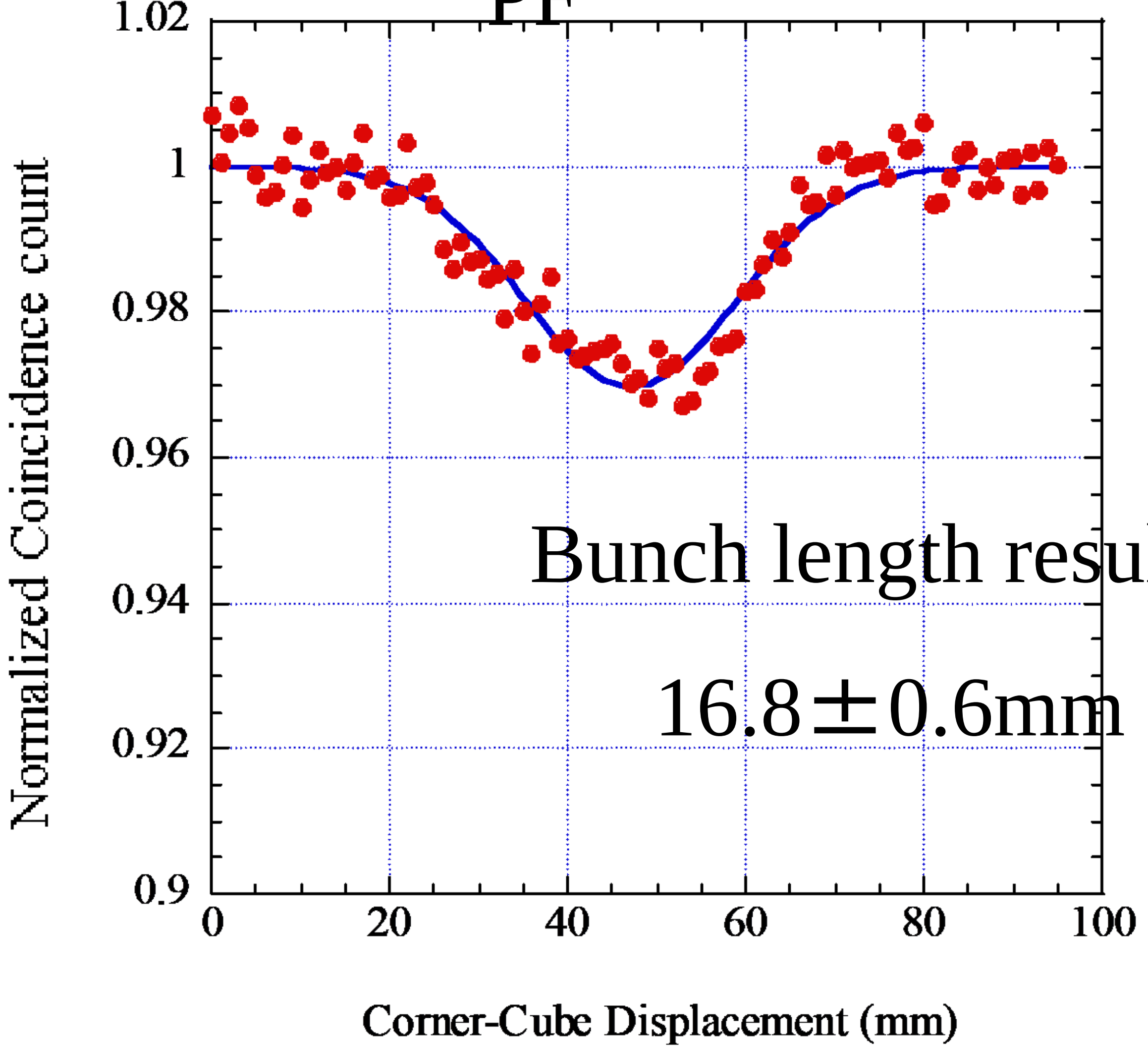


Example of $Count_{12}$

Under collimation



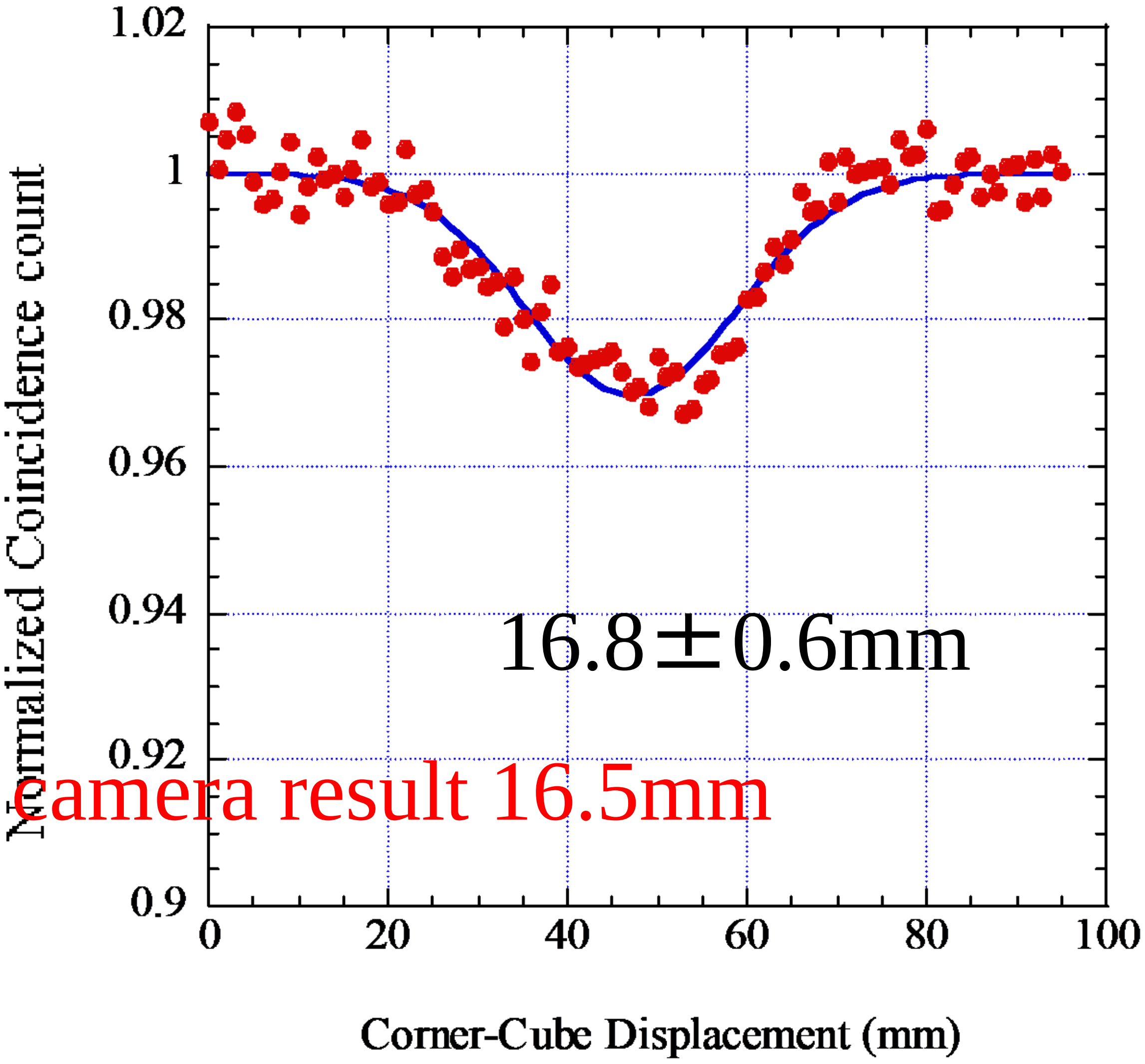
Intensity interferogram at the PF



Bunch length result is

16.8±0.6mm

Bunch length result



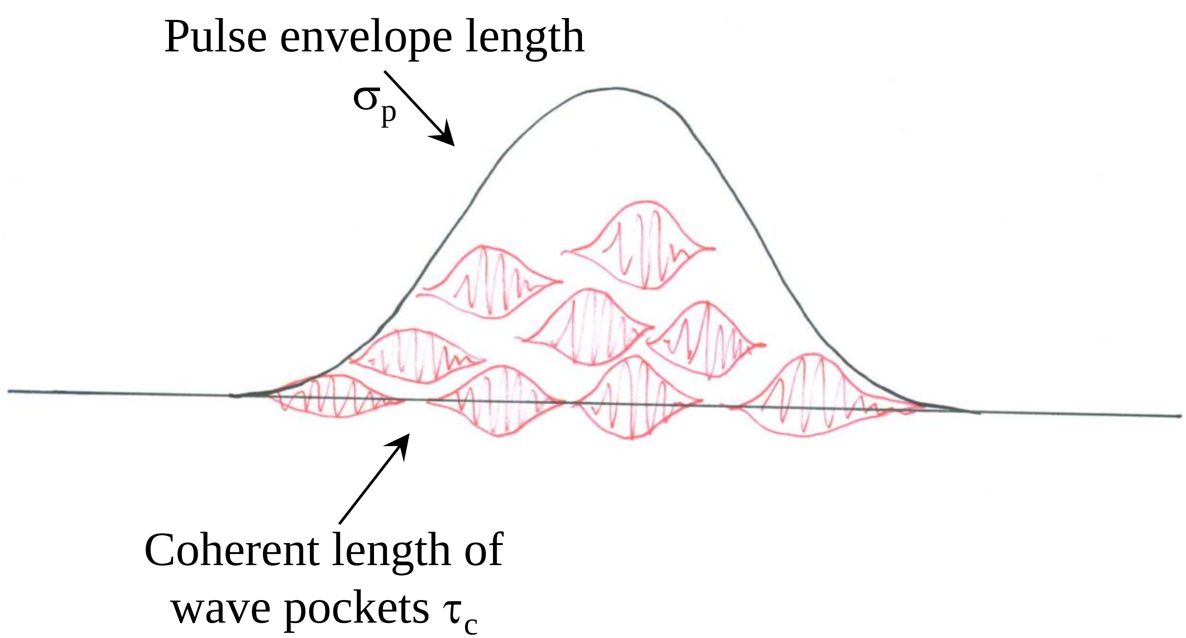
streak camera result 16.5mm

Theoretical temporal resolution of the intensity interferometer

Pulse envelope length σ_p is always longer than Coherent length of wave pockets τ_c.

$$\sigma_p \geq \tau_c$$

We can measure the very short pulse length with intensity interferometry with nearly no theoretical limit on temporal resolution.



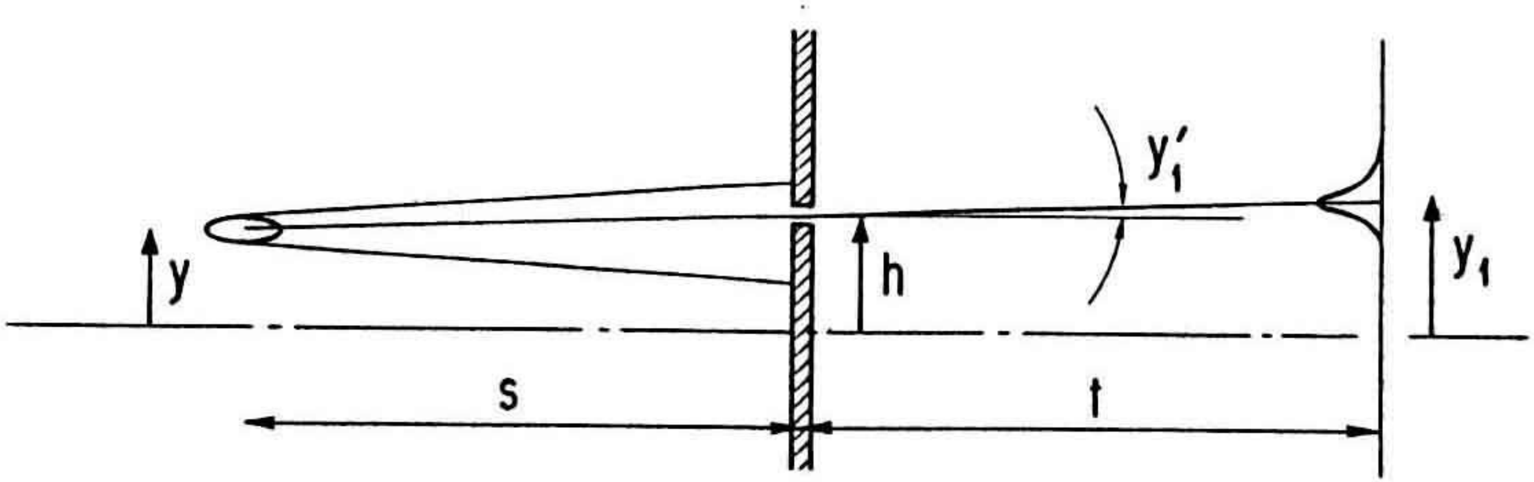
Actual resolution will be limited by dispersion of the glass. <10fsec

Conclusions of intensity interferometry

1. Bending radiation are chaotic as the ensamble of photons.
2. Intensity interferometry is fully applicable to measure very short bunch length.

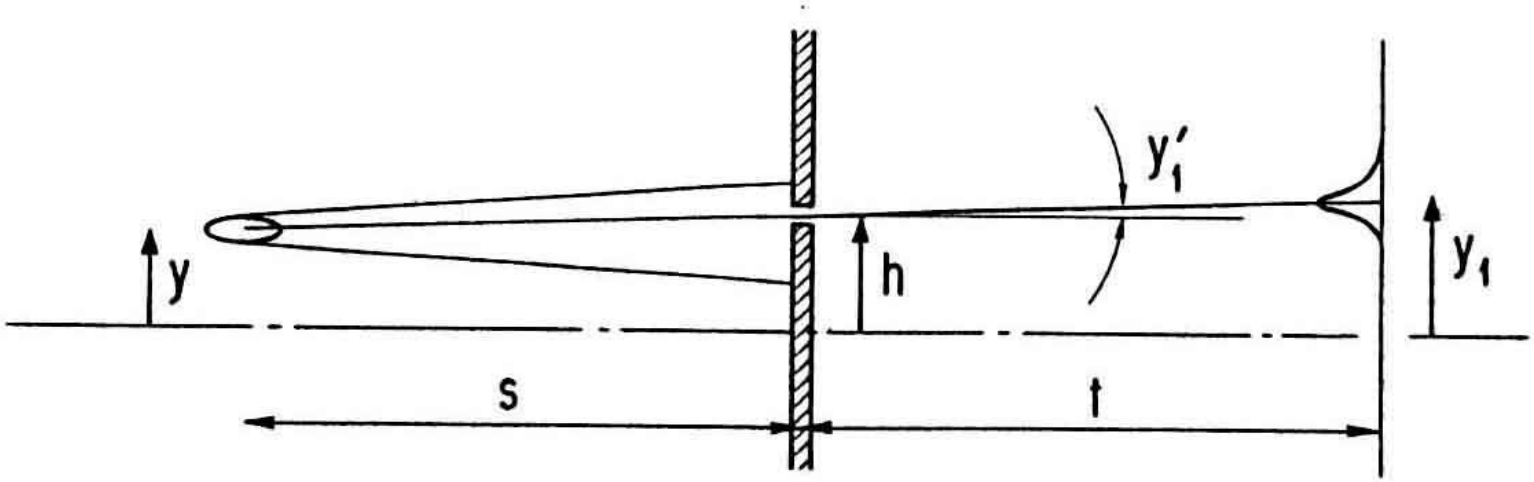
SR monitor based on X-ray

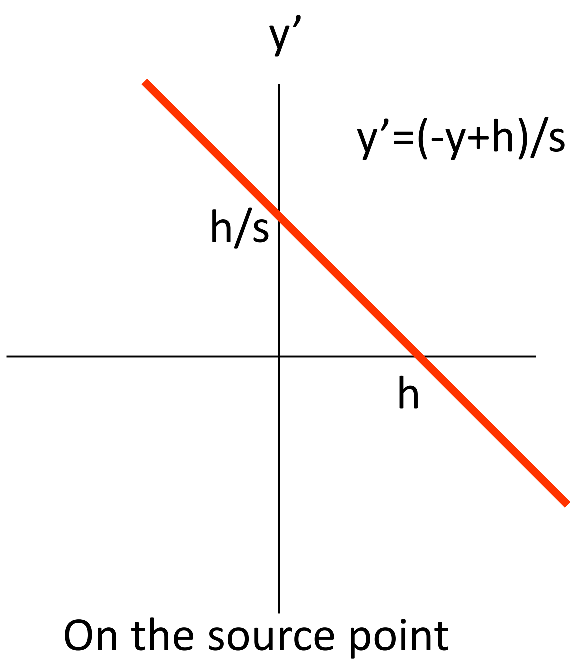
X-ray pin-hole camera

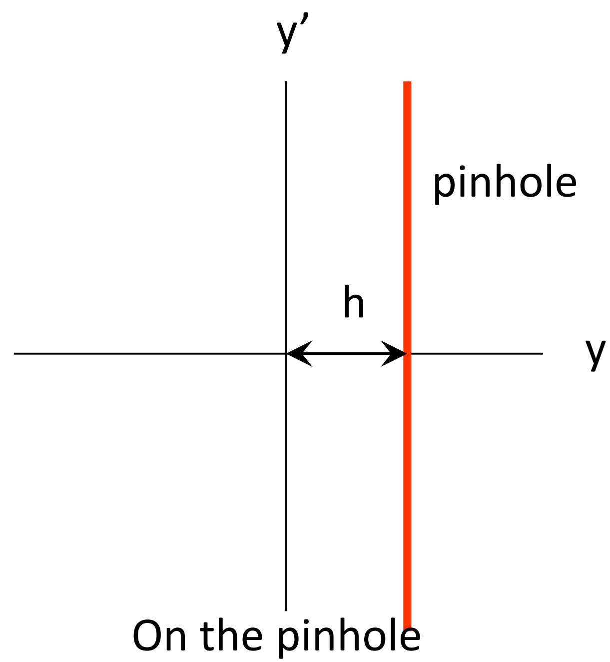


$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} 1 & s+t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} .$$

Representation in the phase space





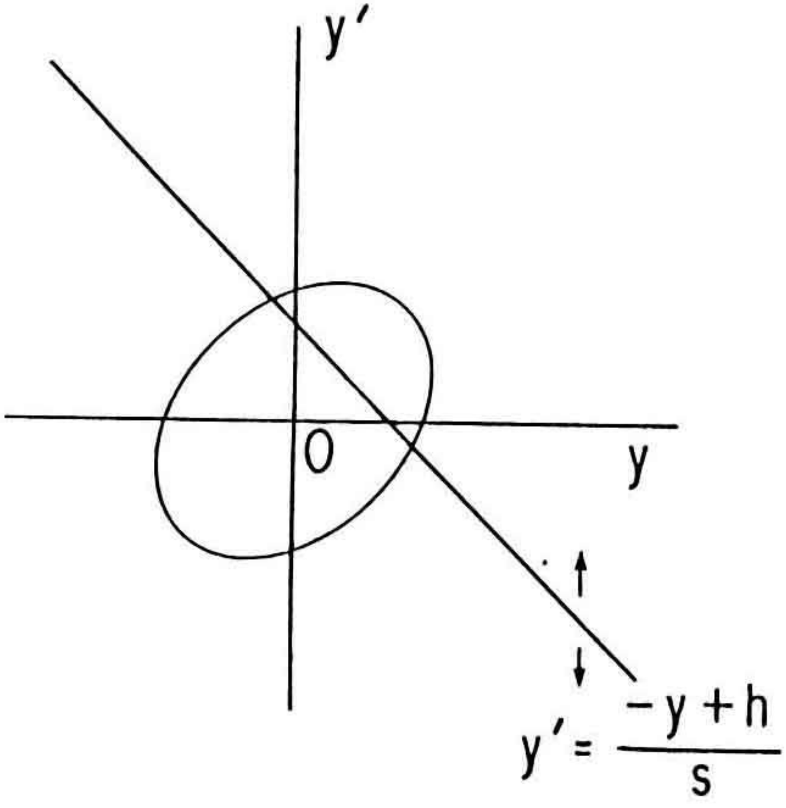


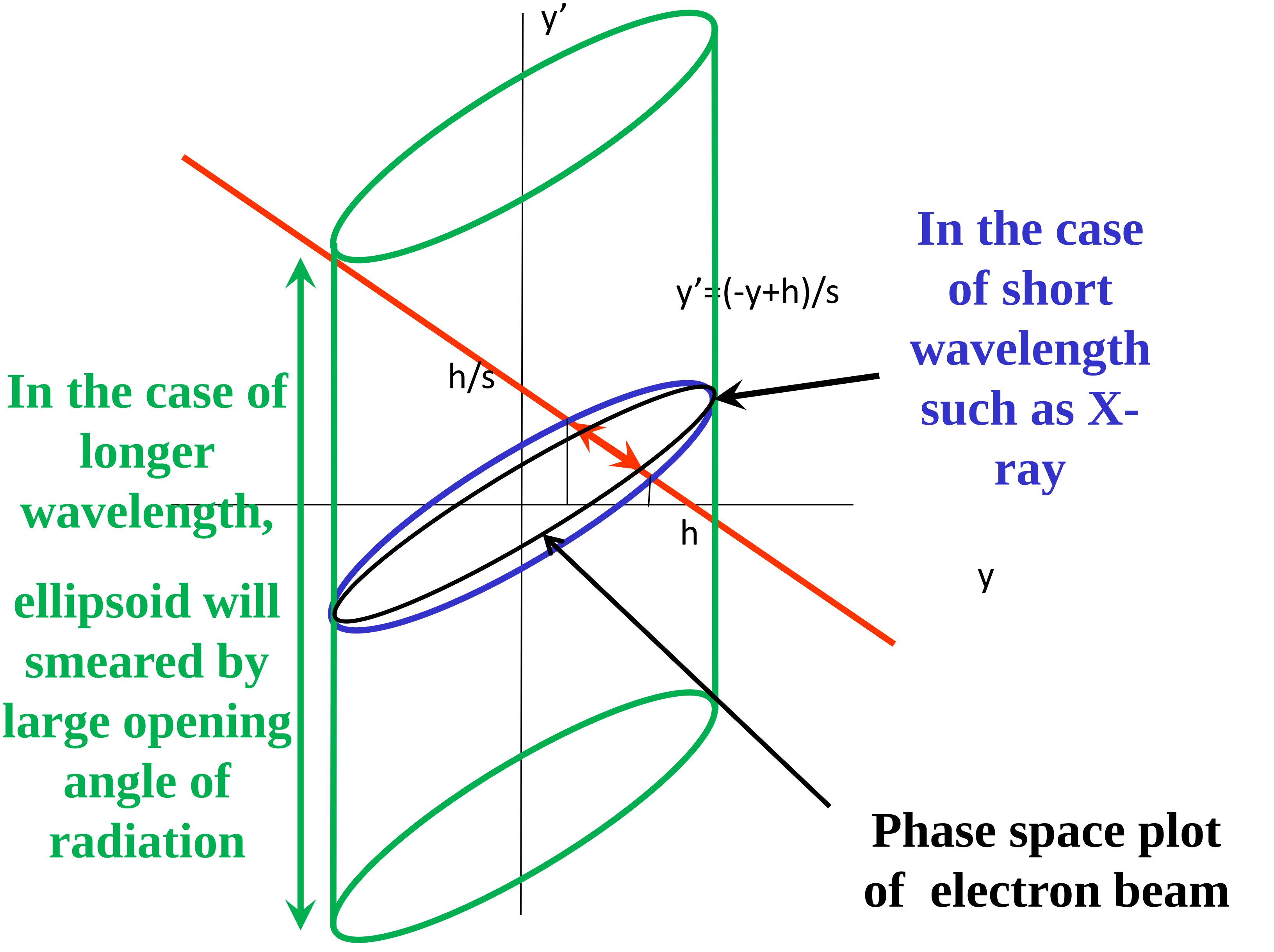
Pinhole in phase space on source point

$$\begin{pmatrix} h \\ y'_p \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$h = y + sy'$$

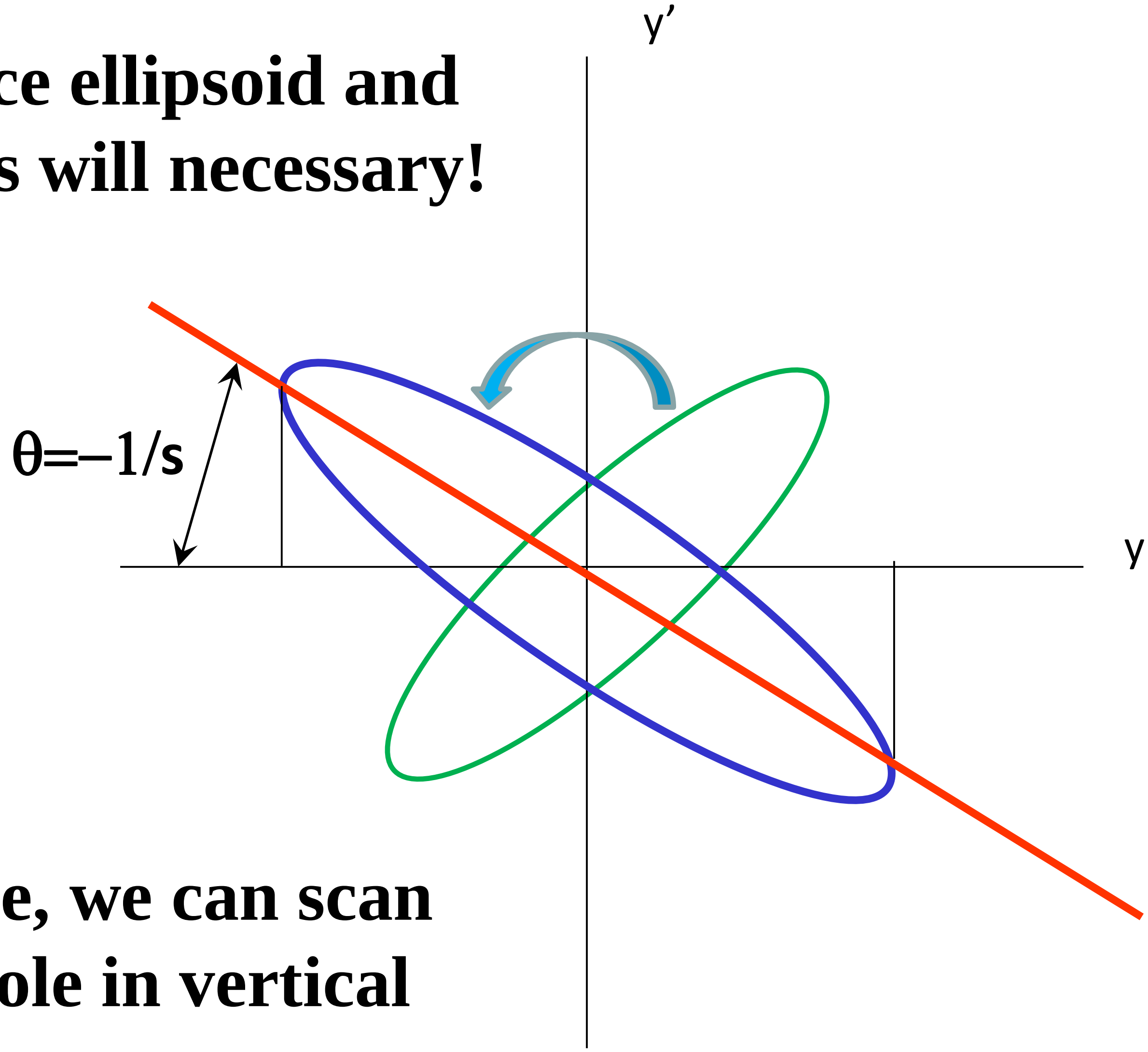
$$y' = (-y+h)/s$$



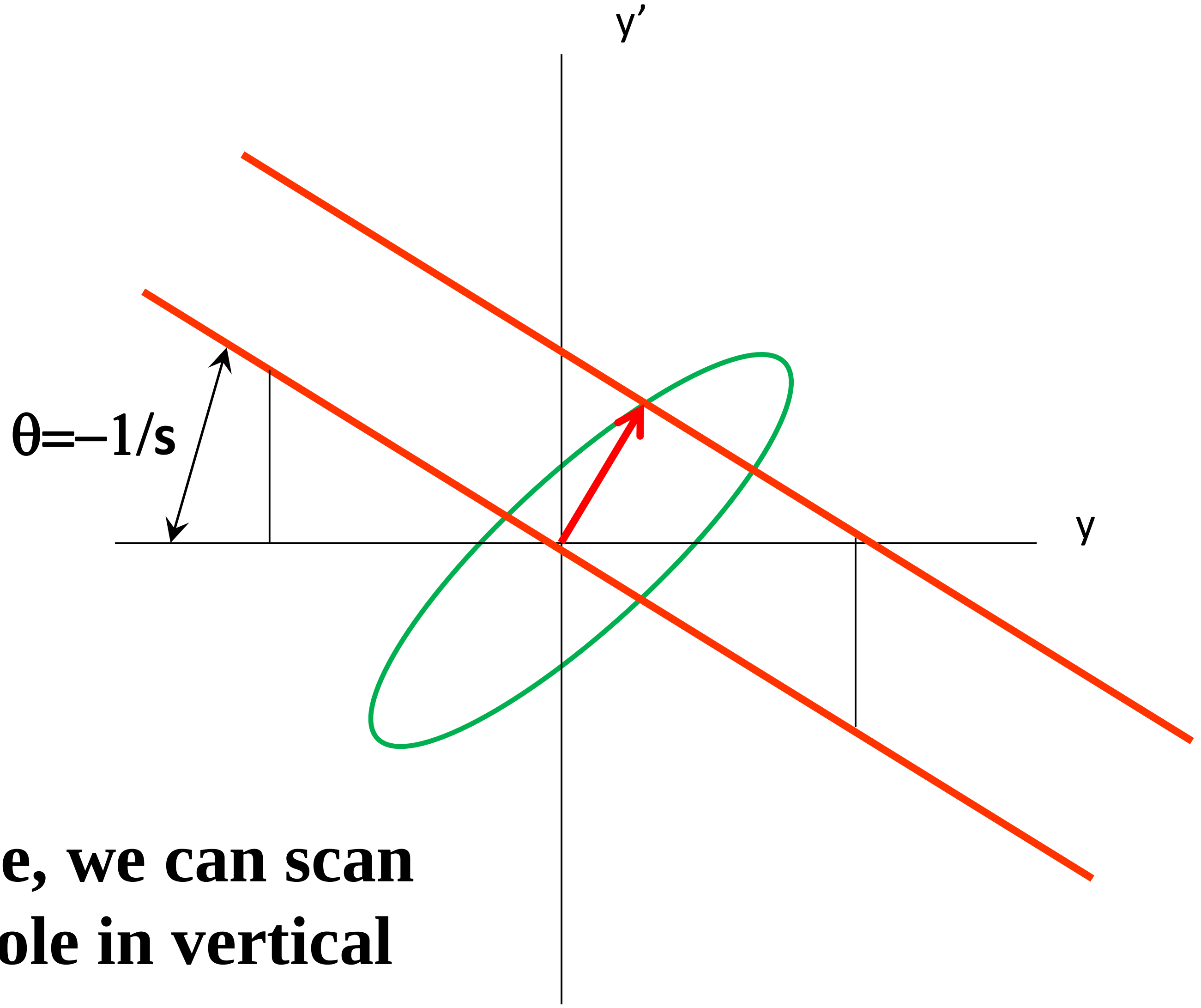


y'
y'=(-y+h)/s
h/s
h
y
In the case of longer wavelength,
ellipsoid will smeared by large opening angle of radiation
In the case of short wavelength such as X-ray
Phase space plot of electron beam

Optimization between emittance ellipsoid and distance s will necessary!

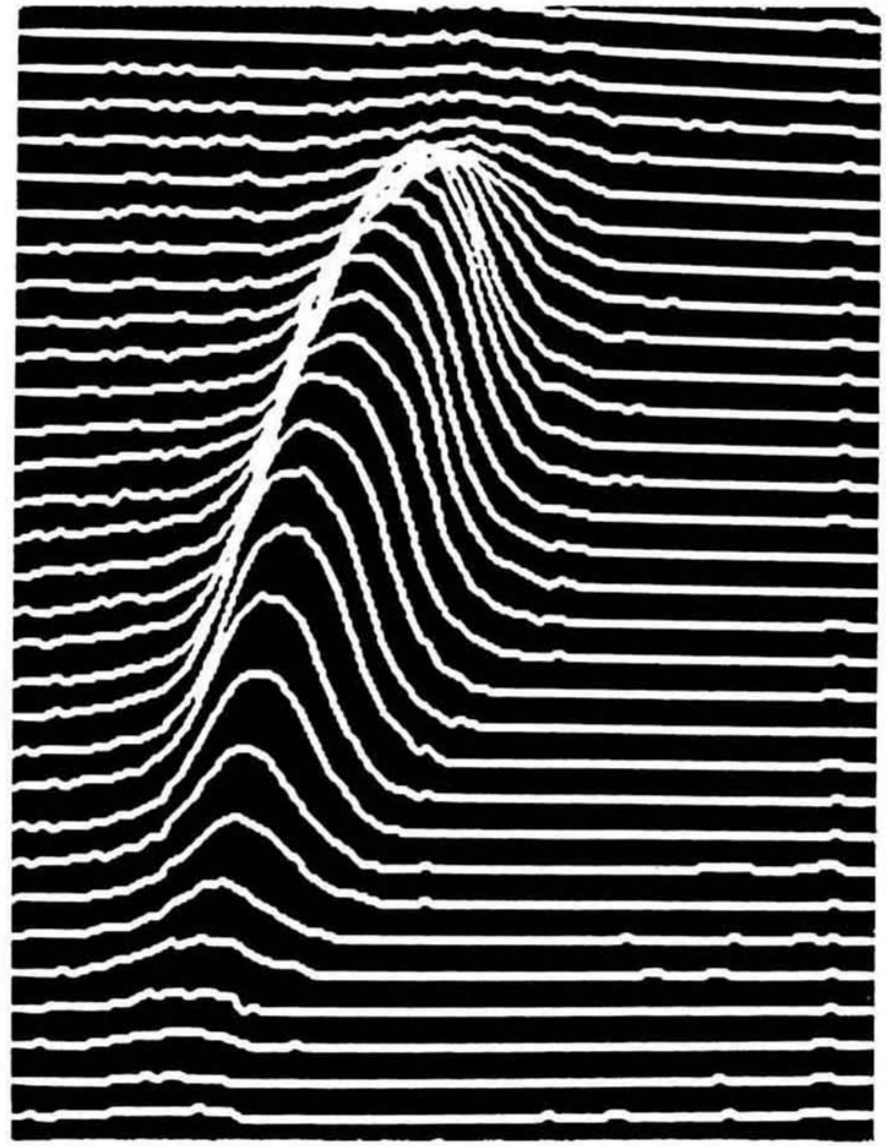


Otherwise, we can scan the pinhole in vertical

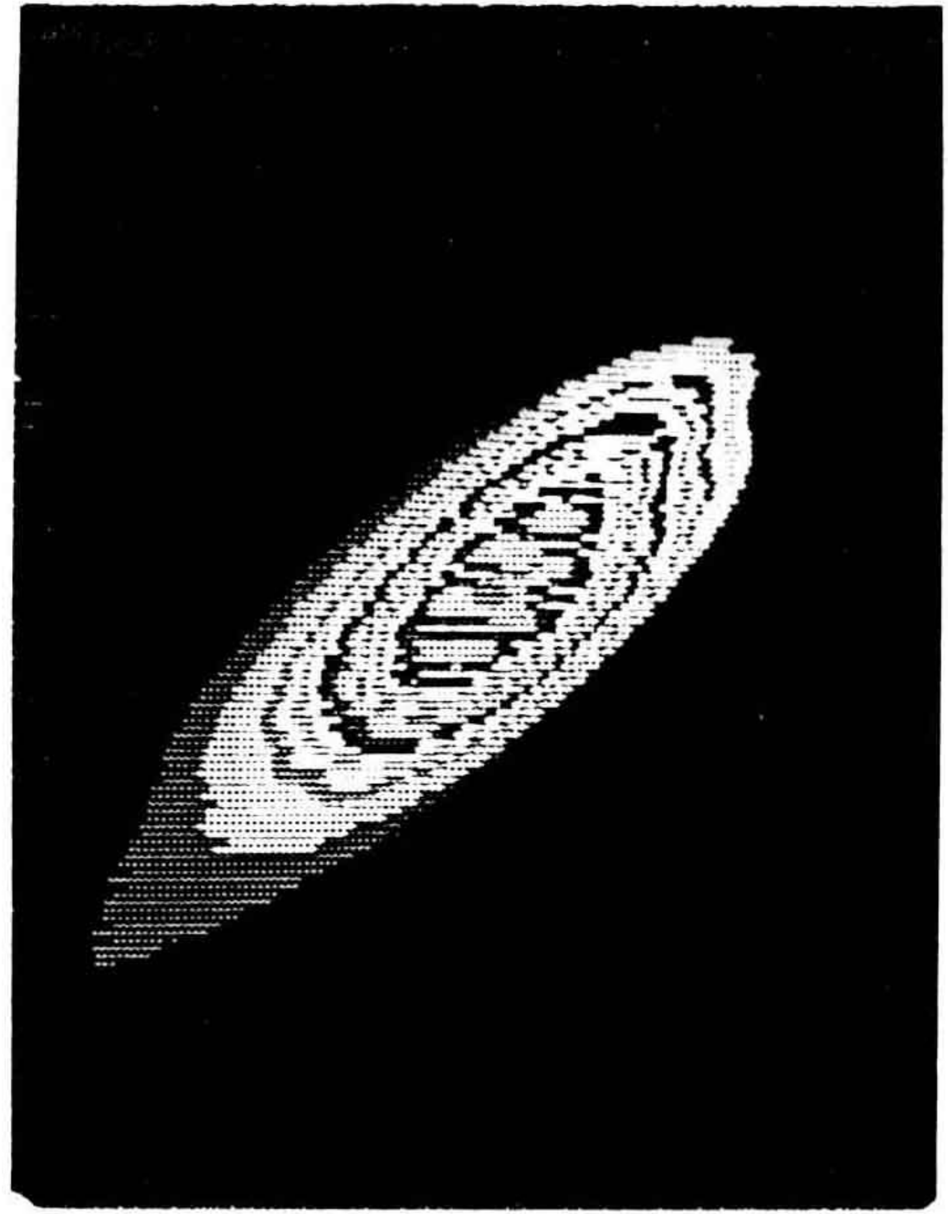


Otherwise, we can scan the pinhole in vertical

Vertical histogram taken at a scan of pinhole height.



Vertical phase space profile reconstructed from vertical histogram.



Diffraction in the pinhole

Intensity of diffraction is given by Fresnel transform of pupil function F of the pinhole

$$I(x,y)=\left|\iint F(x_0,y_0)\exp\left\{\frac{ik}{2z}\left[(x_0-x)^2+(y_0-y)^2\right]\right\}d\xi d\eta\right|^2$$

Thus, in the case of simple circular hole

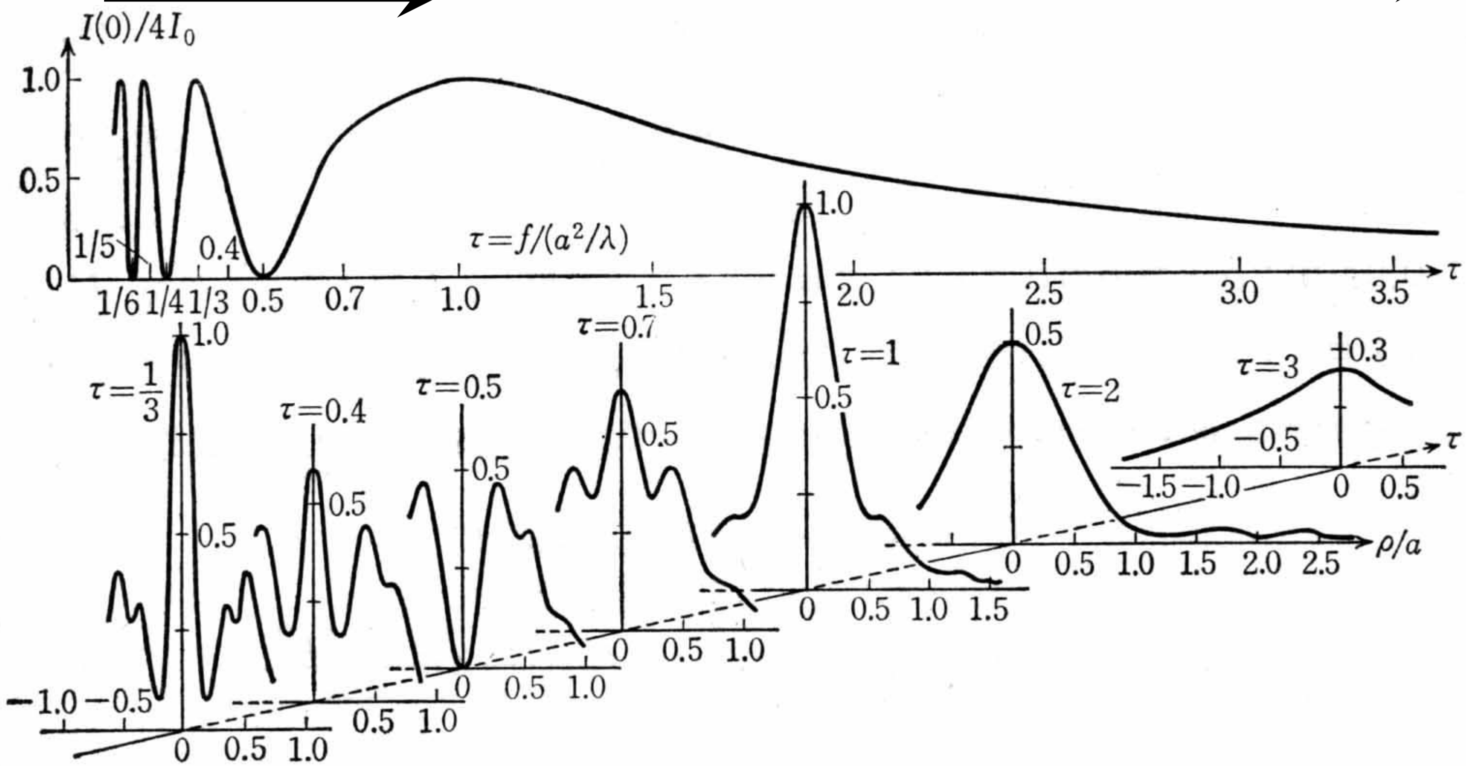
$$I(x,y)=\left|2\pi\int \exp\left\{izr^2\right\}\cdot J_0(Rr)dr\right|^2$$

$\tau = f/(a^2/\lambda) < 1$

Fresnel like region

$\tau = f/(a^2/\lambda) > 3$

Franhofer like region



For example, λ=0.1nm, a=2μm

Franhofer region > 0.12m

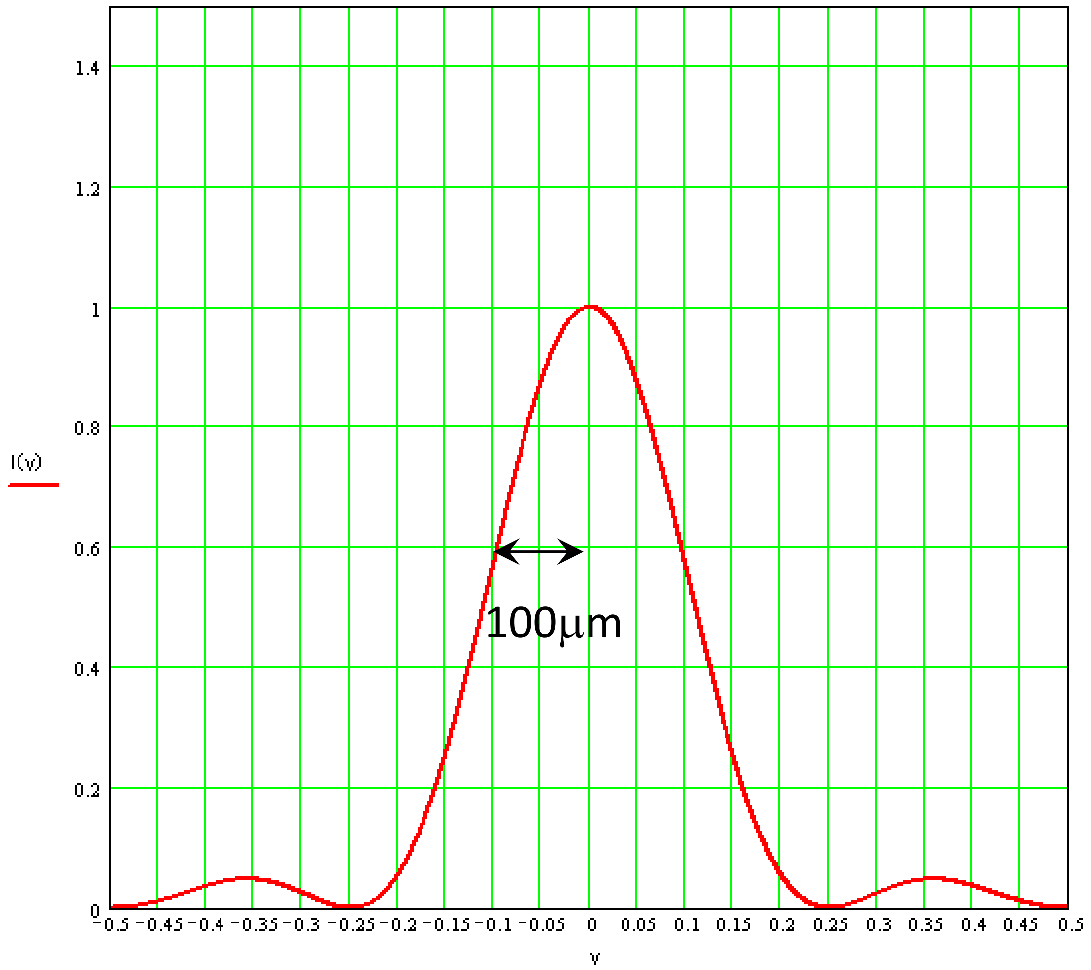
In most case, pinhole with x-rays should be Franhofer region.

λ=0.1nm

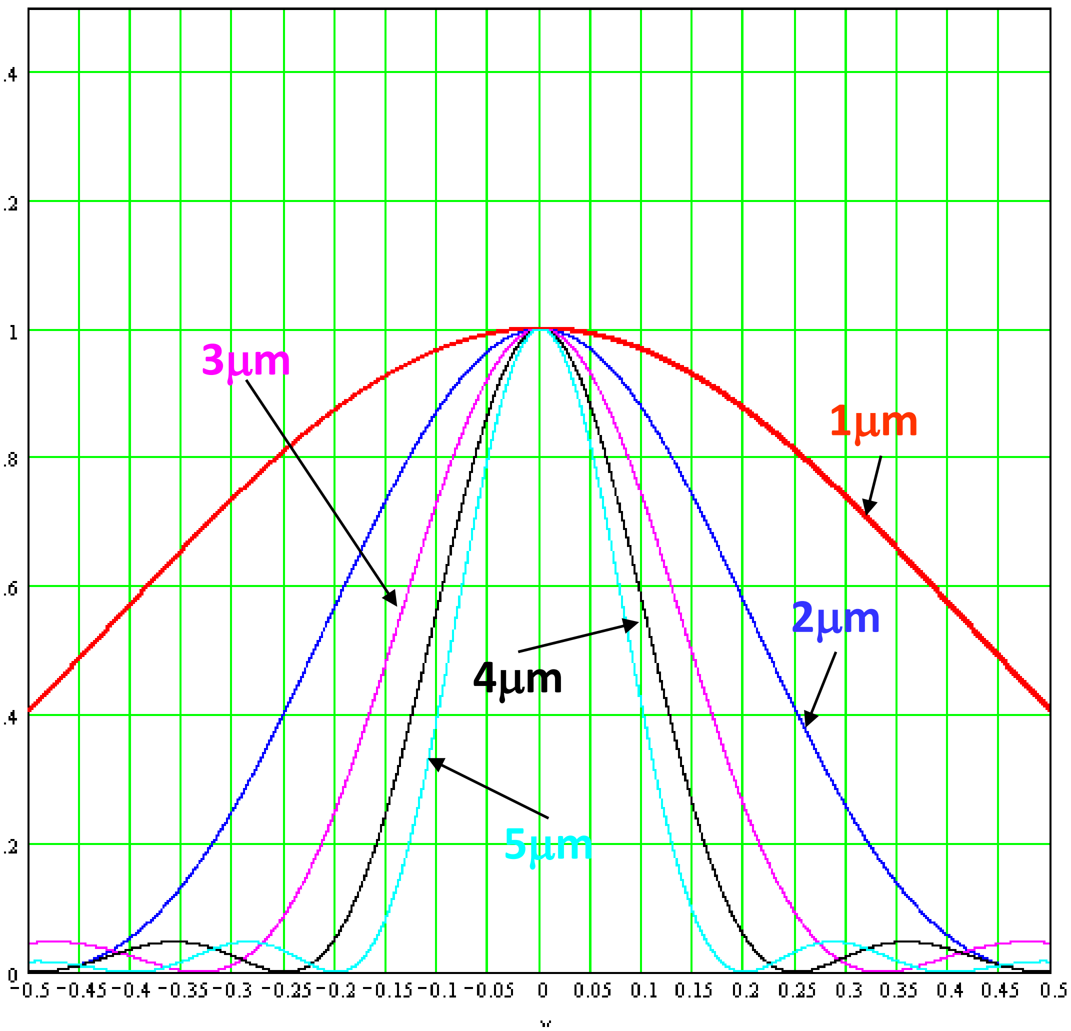
a=2μm

(D=4μm)

F=10m

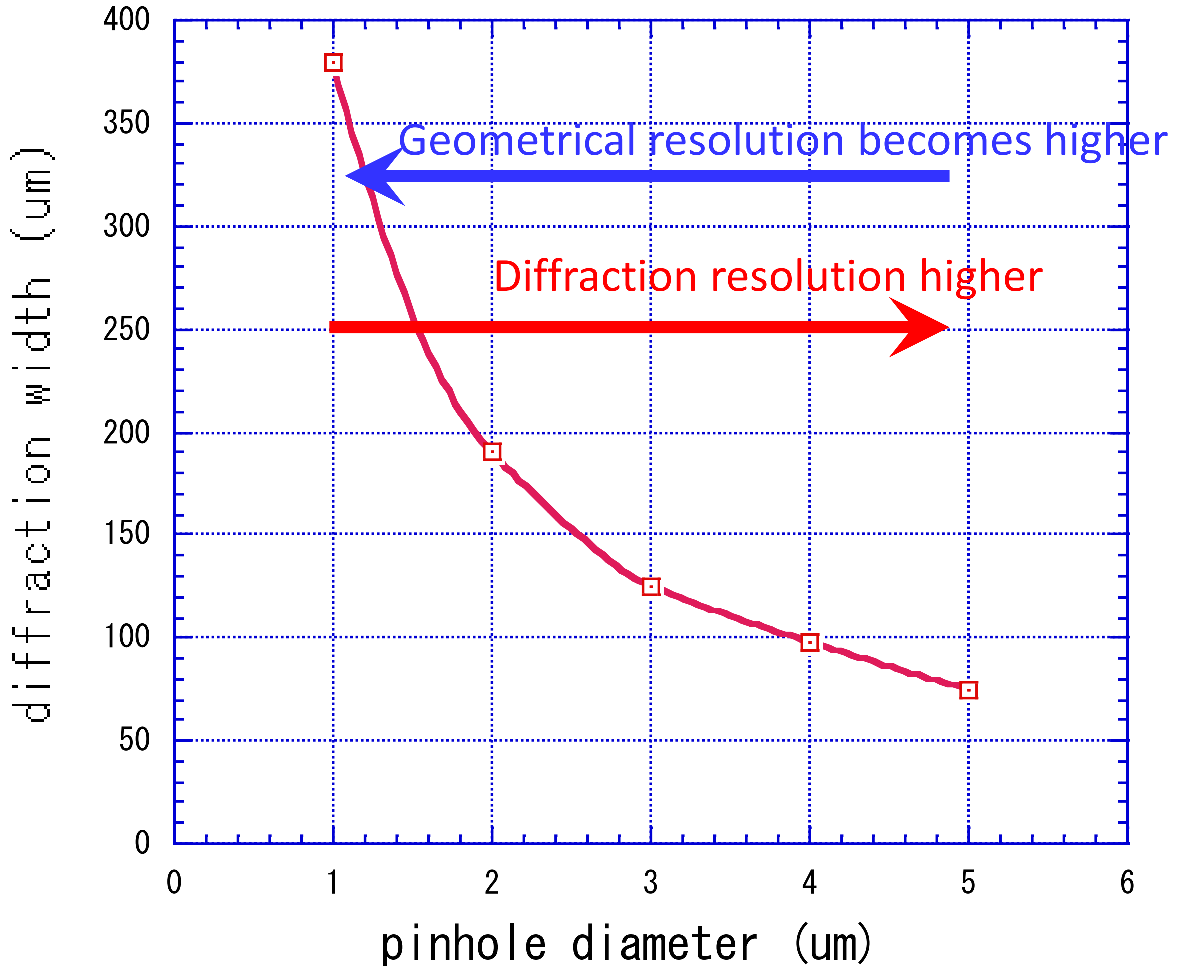


Diffraction patterns for several wave lengths



Diffraction width as a function of pinhole diameter

λ=0.1nm, F=10m



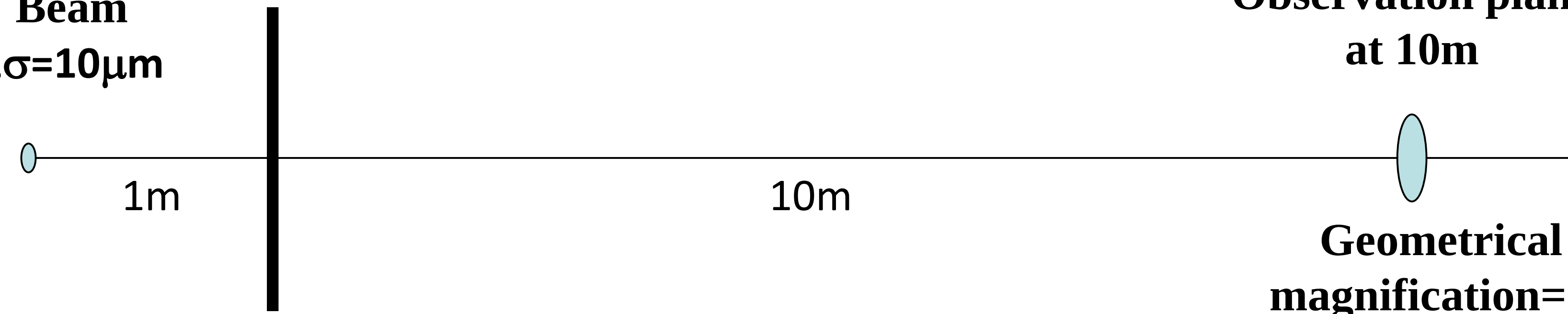
An example of pinhole camera measurement setup

Diffraction size should be 100μm

Pinhole D=4μm

Observation plane at 10m

Beam
1σ=10μm



Geometrical magnification=10

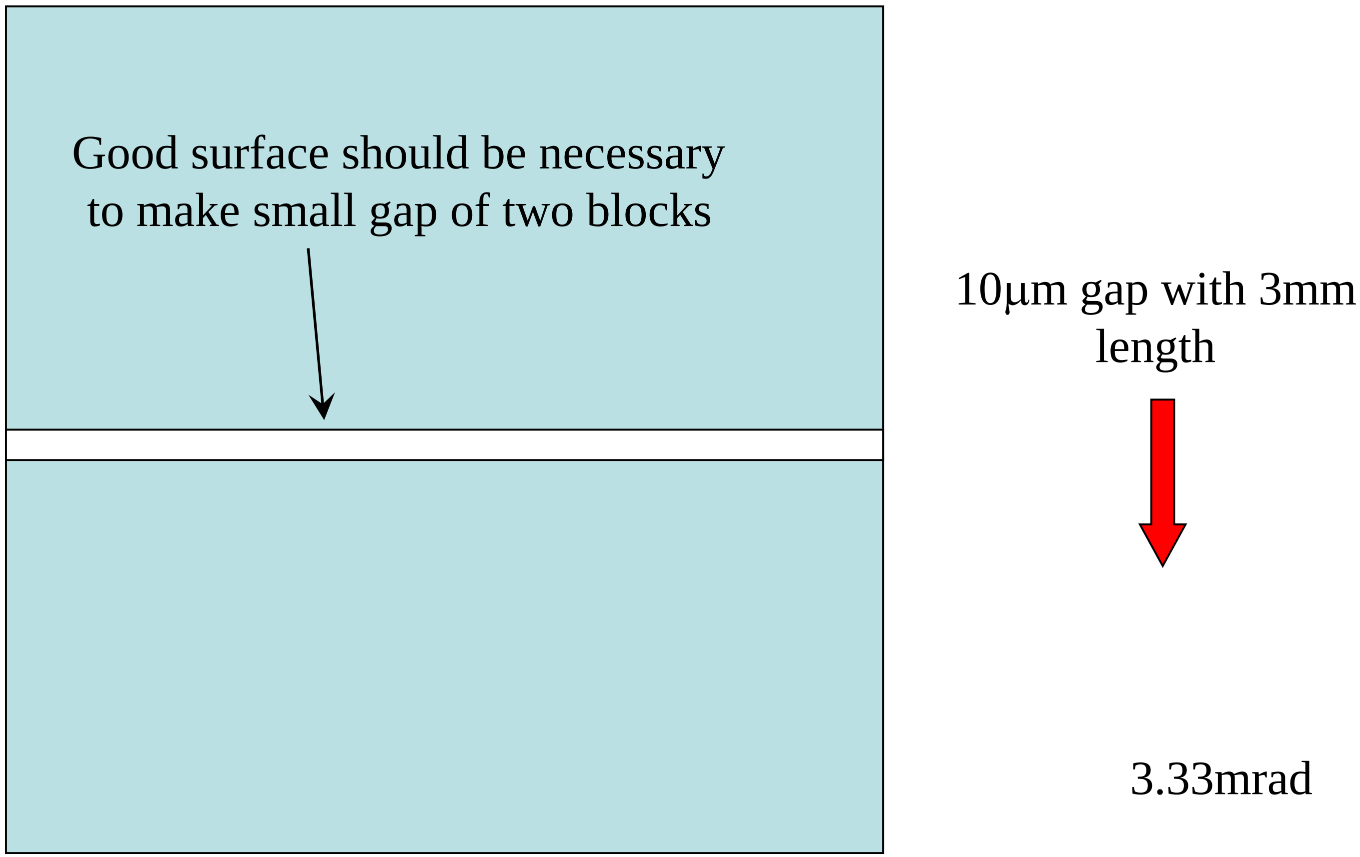
Result of beam size should be 14μm

without convolution of pinhole diameter

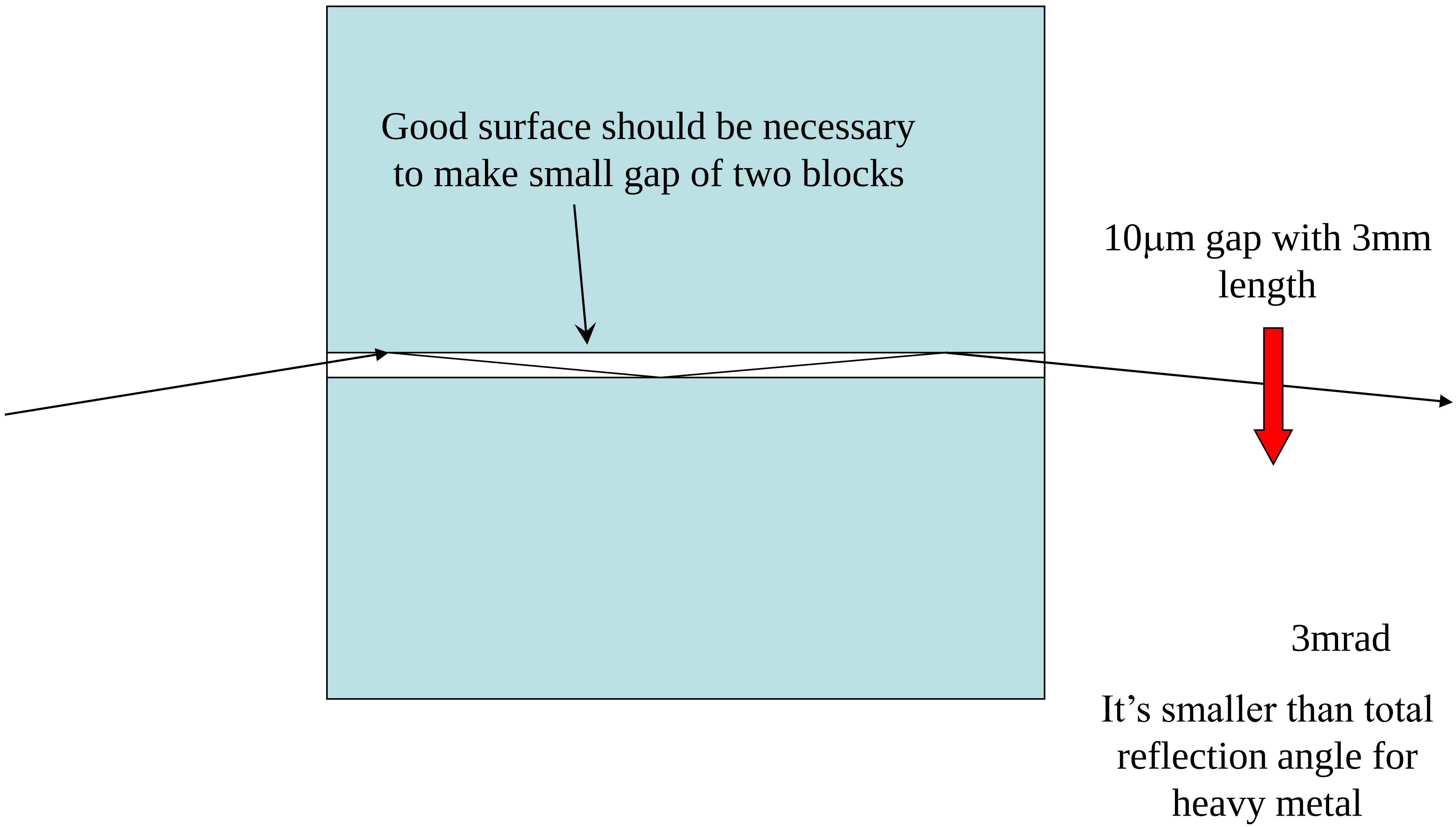
Geometrical image size should be 100μm

Without convolution between pinhole diameter

Total reflection by surface of pinhole blocks



Totally reflection by surface of pinhole blocks

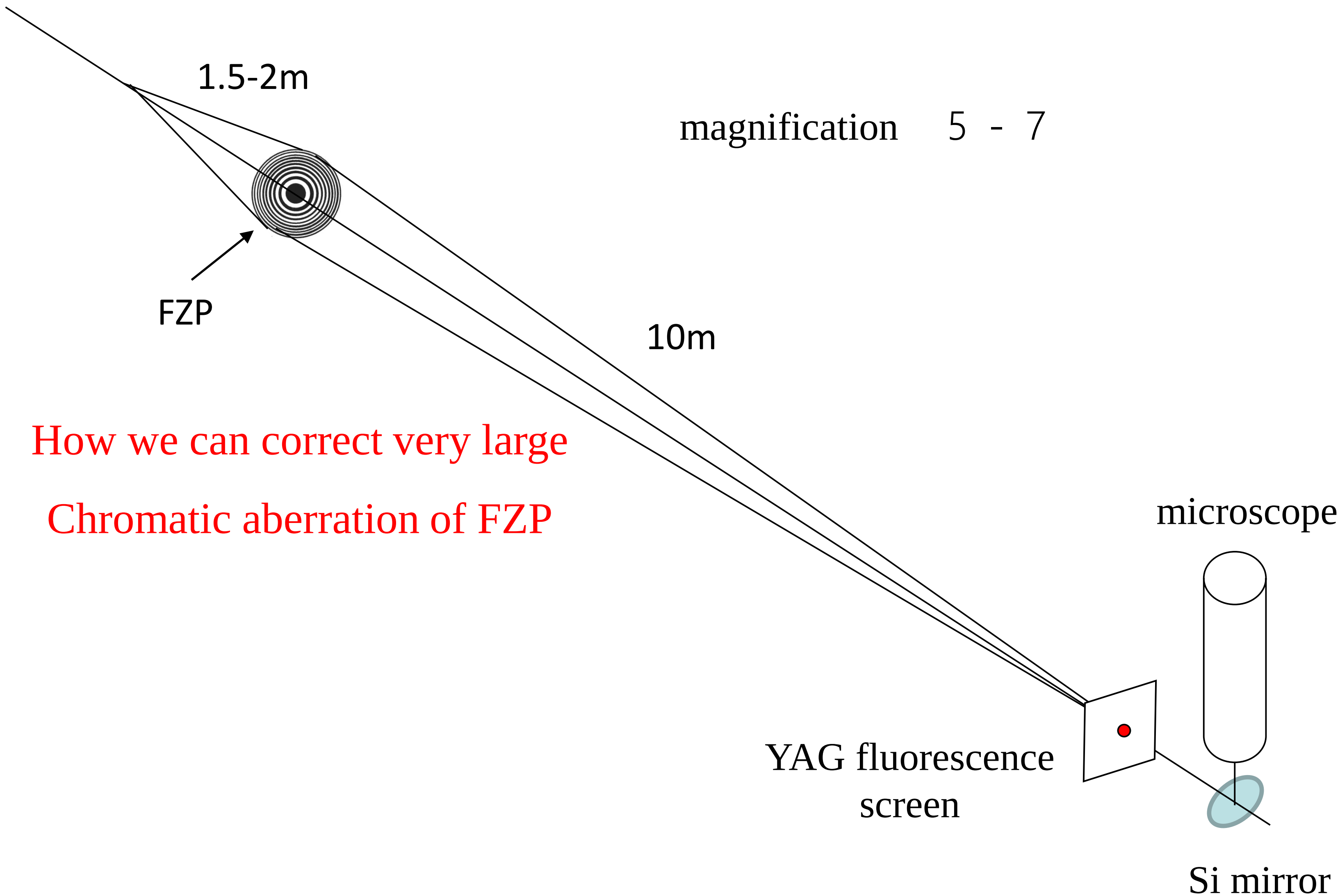


From these evidences,

decomvolution by PSF including many effects such as diffraction, total reflection and halation in detection system etc.

Decomvolution is very important in X-ray pin-hole camera.

X-ray Fresnel zone plate

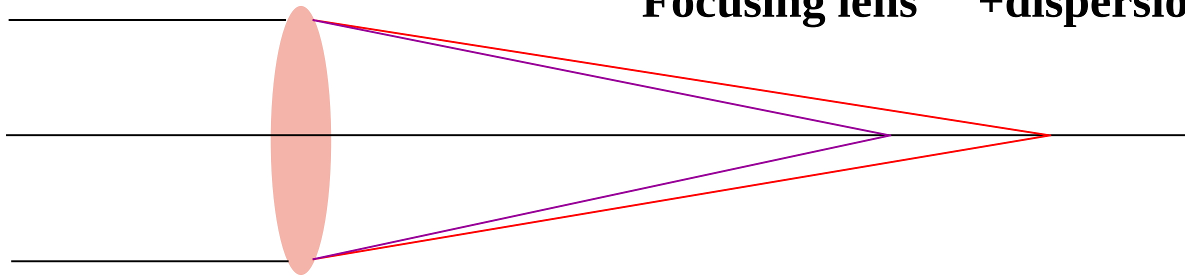


How we can correct very large

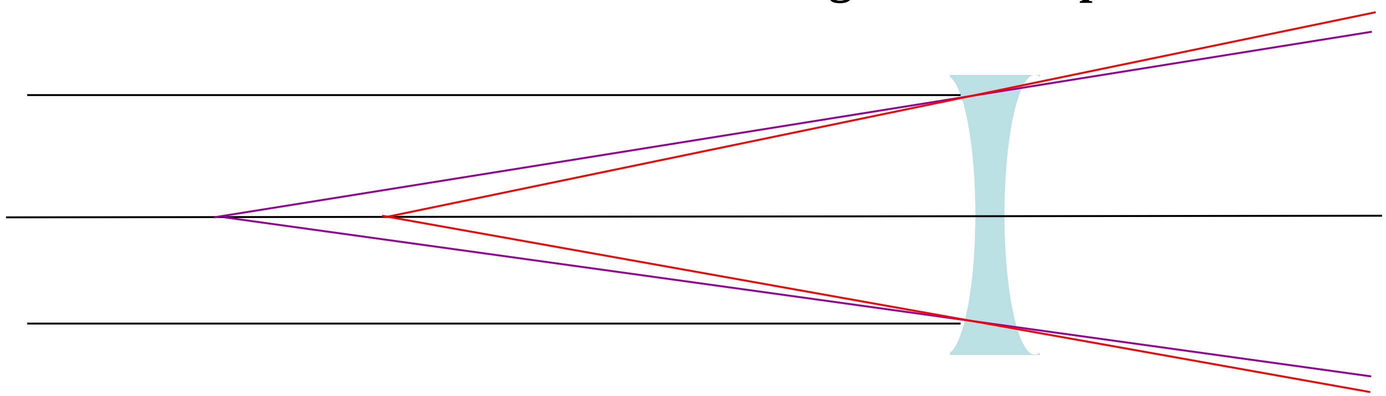
Chromatic aberration of FZP

Concept of achromatic lens

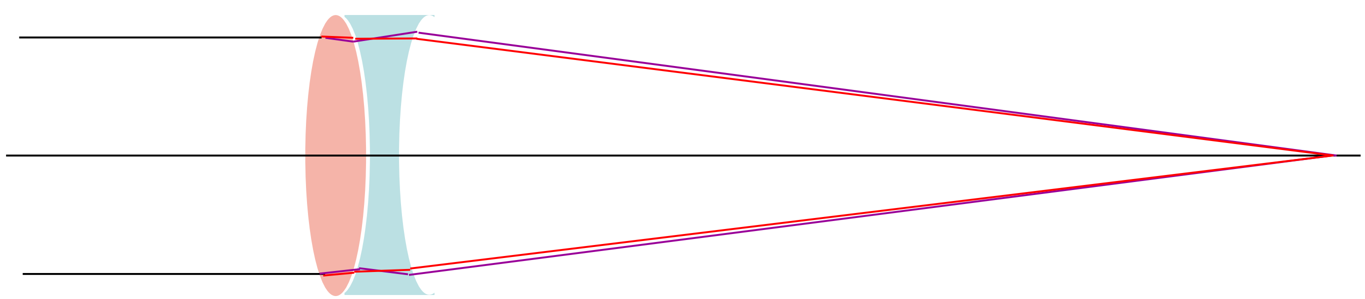
Focusing lens +dispersion



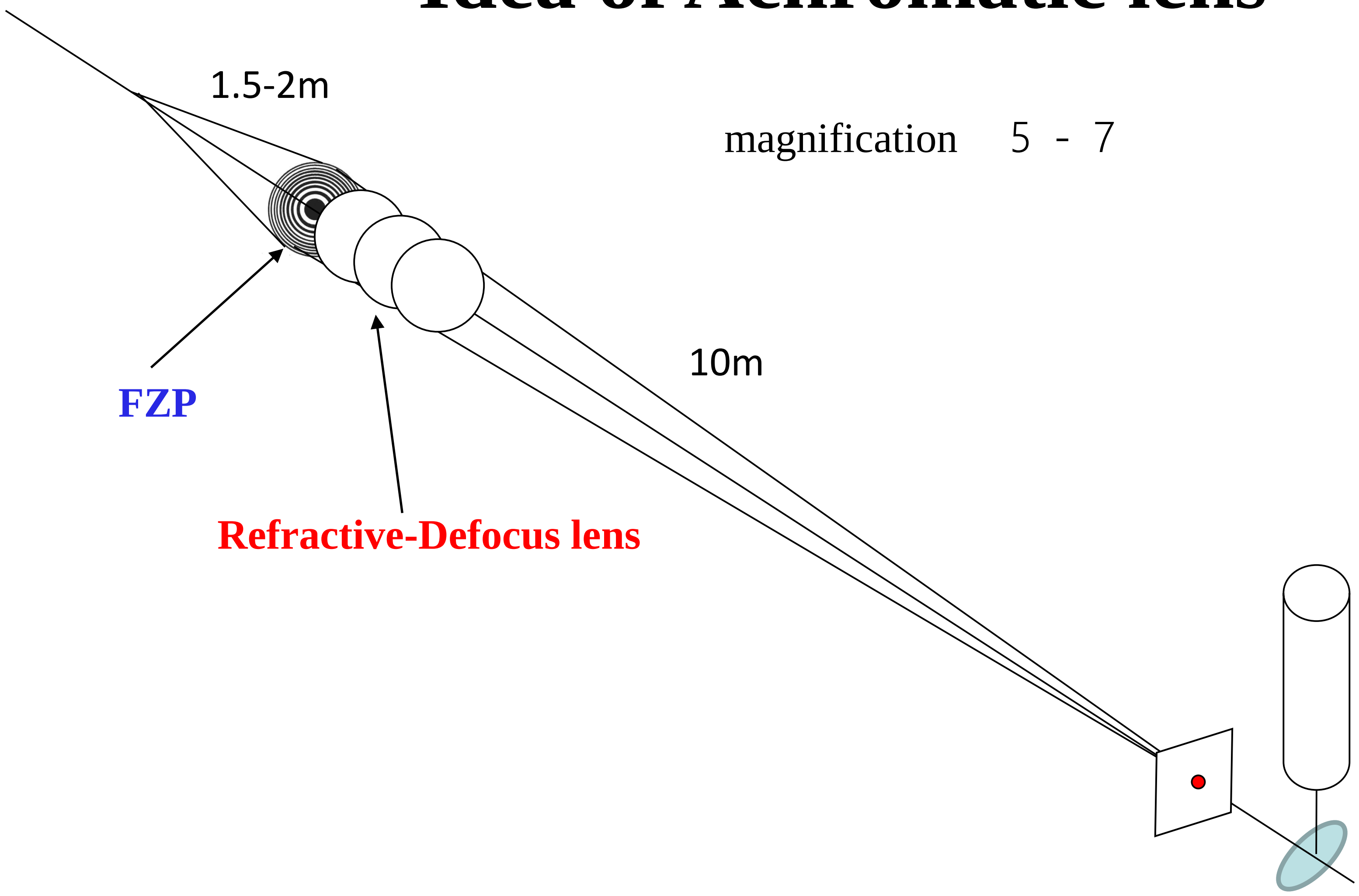
Defocusing lens -dispersion



Certain combination of focusing and defocusing lens can cancel focal shift



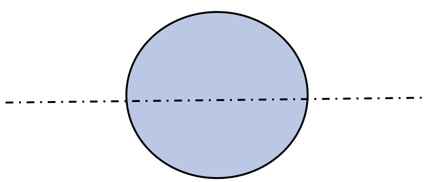
Idea of Achromatic lens



FZP + refractive lens

(a)

(b)



Abbe number of FZP

About -8 very high dispersion

Abbe number

X-ra defocusing lens

About 0.8 extremely high dispersion

Combination of these two components, we can make achromatic lens with focusing length of 1m.

f=1m

FZP: 0.9m

XDL: -9m

Decomvolution with OTF

Decomvolution using the Wiener inverse filter

Fourier transform of blurred image $G(f_x,f_y)$ in spatial frequency domain (f_x,f_y) is given by,

$$G(f_x,f_y)=\mathscr{H}(f_x,f_y)F(f_x,f_y) + N(f_x,f_y)$$

where $\mathscr{H}(f_x,f_y)$ is thought as a inverse filter (Fourier transform of PSF), $F(f_x,f_y)$ is a Fourier transform of geometric image, and $N(f_x,f_y)$ is a Fourier transform of noise in the image). So, inverse problem can be written by,

$$F(f_x,f_y)=\frac{G(f_x,f_y)-N(f_x,f_y)}{\mathscr{H}(f_x,f_y)}$$

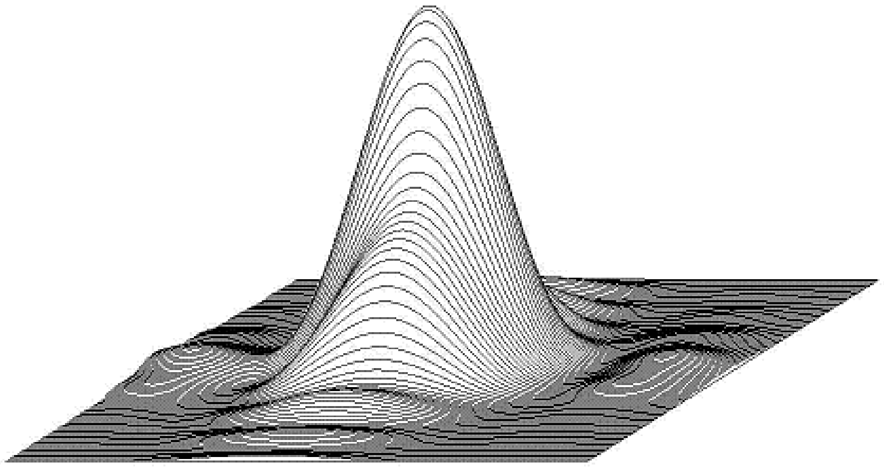
Winner introduce so-called winner's inverse filter;

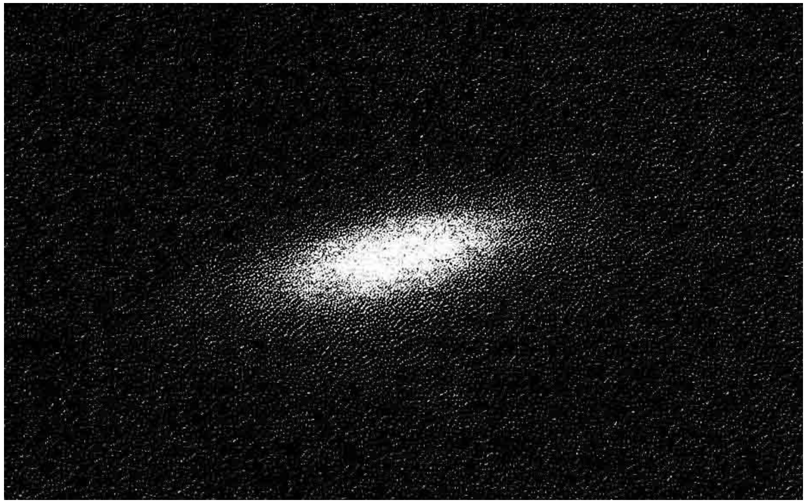
$$H_w(f_x, f_y) = \frac{\mathscr{H}^*(f_x, f_y)}{\left| \mathscr{H}(f_x, f_y) \right|^2 + \dfrac{\phi_n(f_x, f_y)}{\phi_f(f_x, f_y)}}$$

where asterisk indicates the complex conjugate of $\mathscr{H}$, and ϕ_n is a power spectra of the noise, and ϕ_f is a power spectra of the signal.

PSF to evaluate inverse filter for decomvolution

Original image of beam





Original image

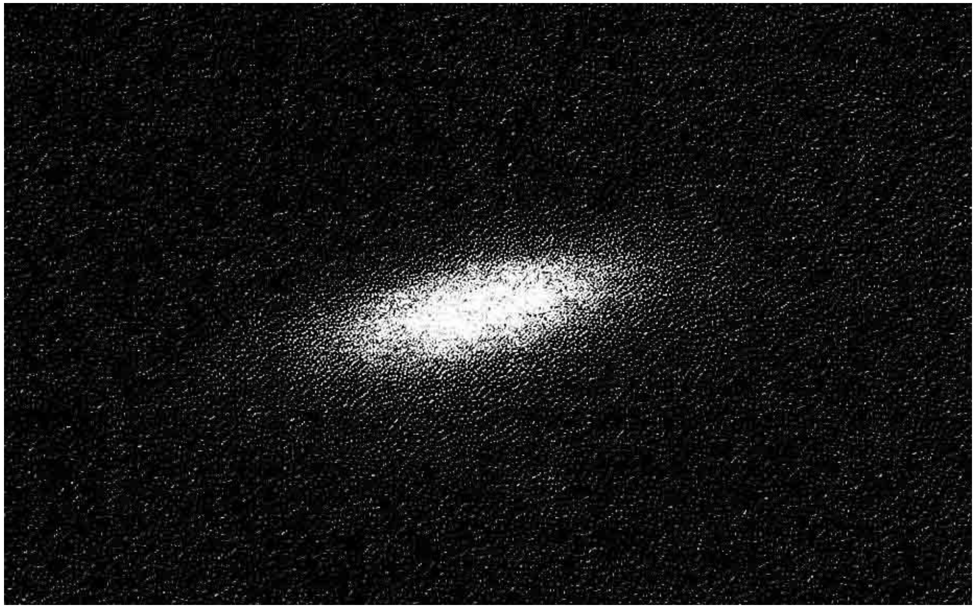
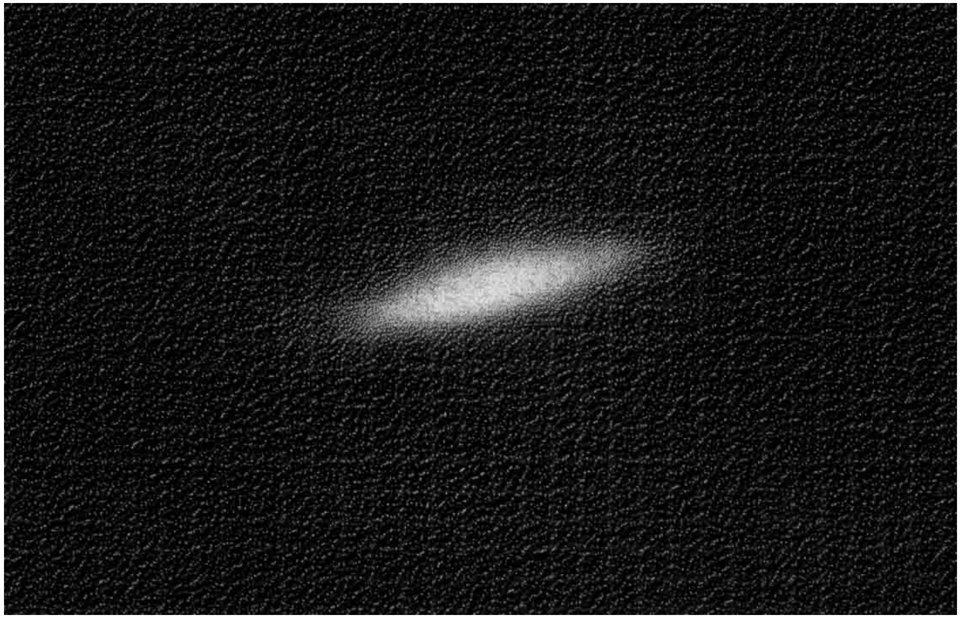
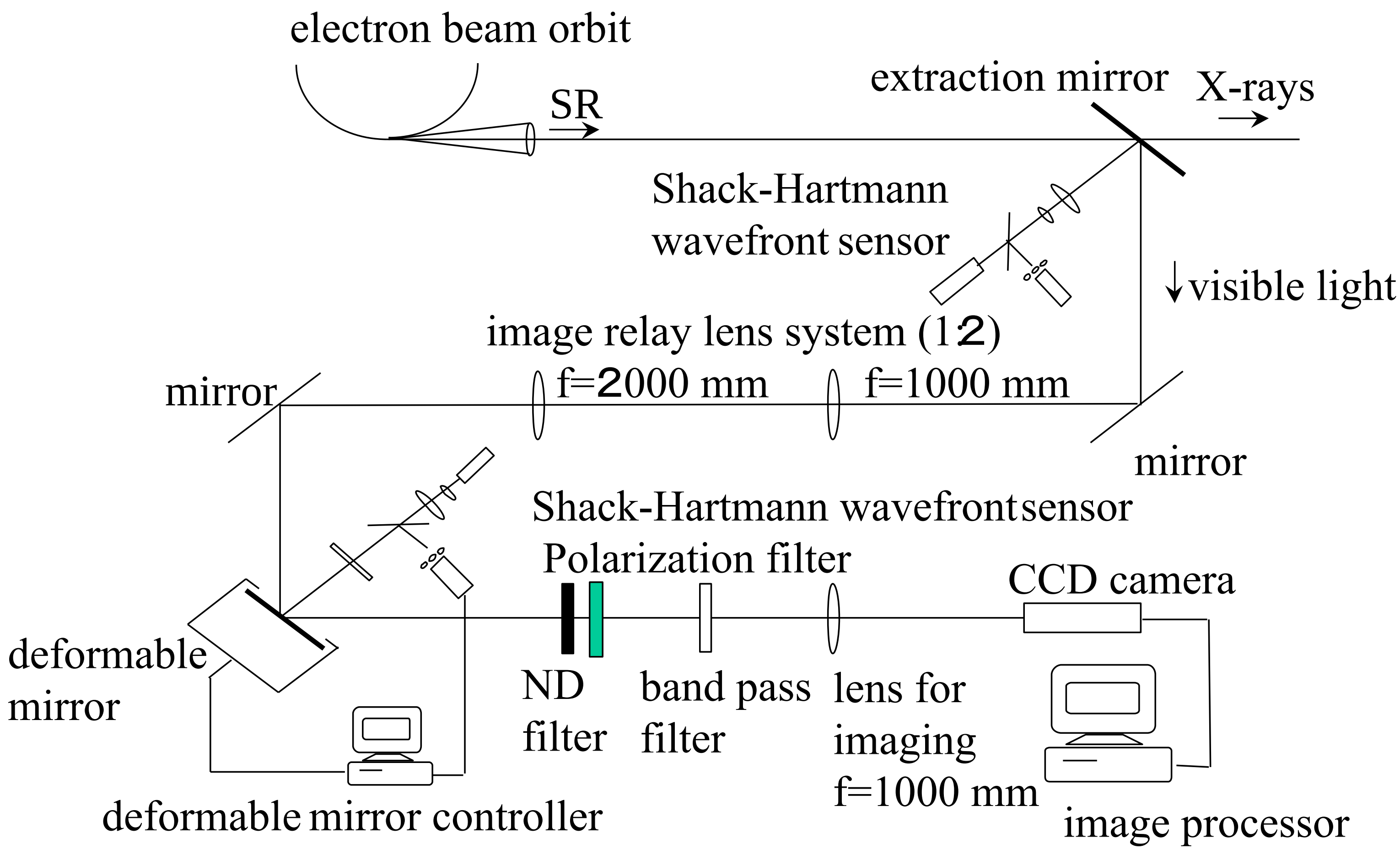


Image after decomvolution

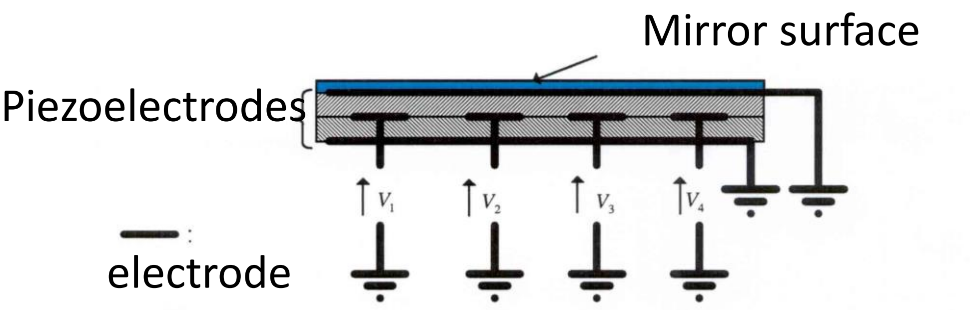


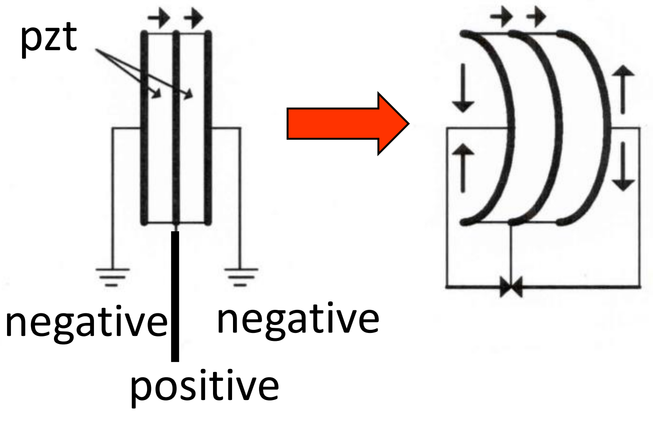
Application of Adaptive optics to correct thermal deformation of extraction mirror


electron beam orbit
SR
extraction mirror
X-rays
Shack-Hartmann wavefront sensor
visible light
image relay lens system (1:2)
f=2000 mm
f=1000 mm
mirror
mirror
Shack-Hartmann wavefront sensor
Polarization filter
CCD camera
deformable mirror
ND filter
band pass filter
lens for imaging f=1000 mm
deformable mirror controller
image processor

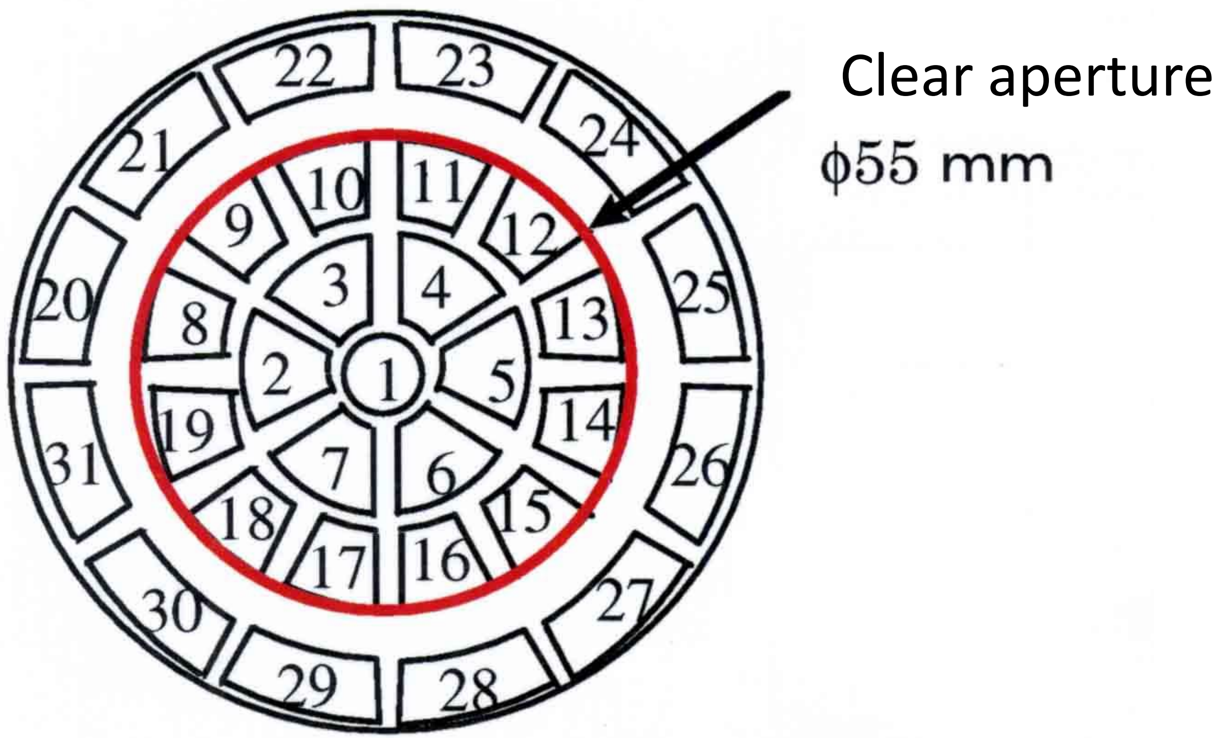
Deformable mirror

Bimorph corrector mirror



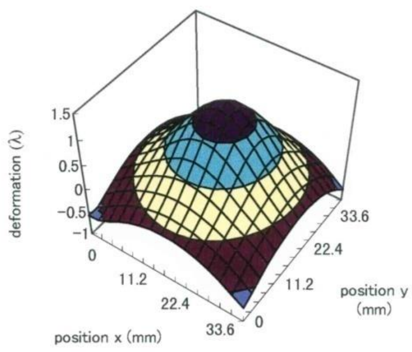


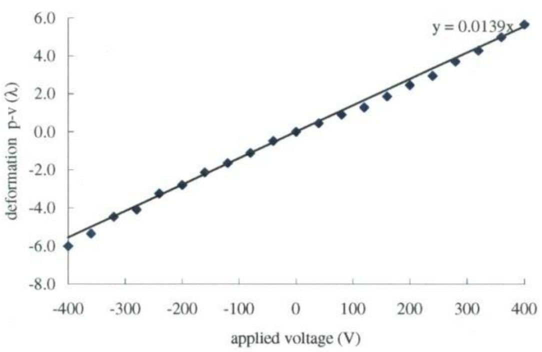
Arrangement of pzt



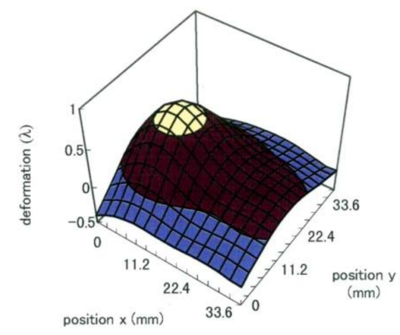
Response of deformation

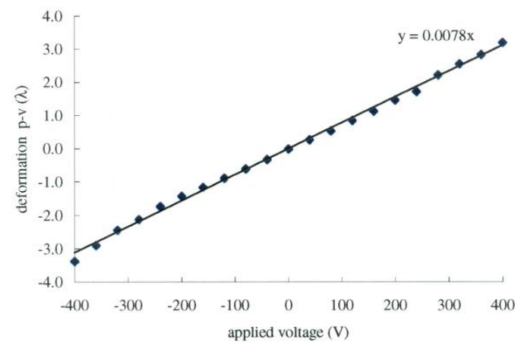
Electrode No.1





Electrode No.2





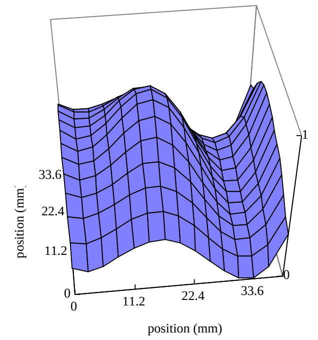
example of wavefront correction at beam current 92mA.

A: wavefront by SR extraction mirror,

B: wavefront by deformable mirror,

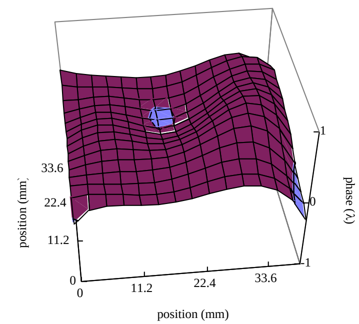
C: summation of A and B.

A



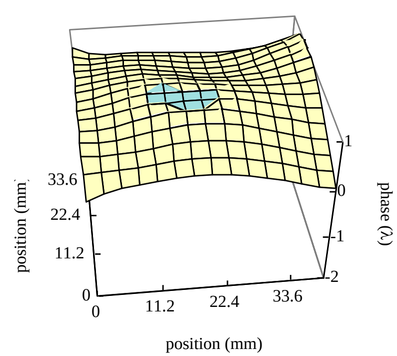
p-v : 0.91l
rms. : 0.21l

B



p-v : 0.92l
rms. : 0.17l

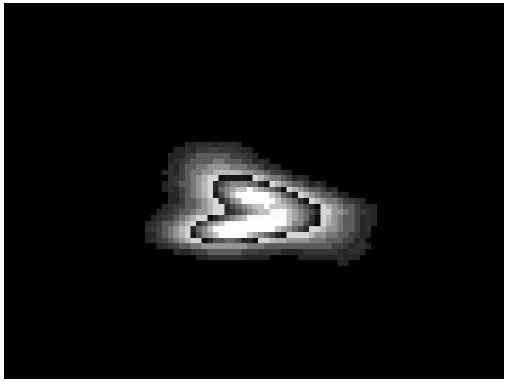
C



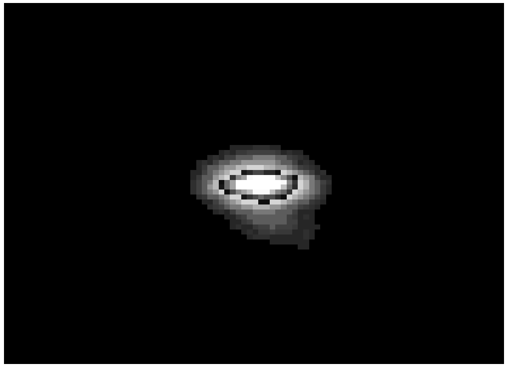
p-v : 1.0 l
rms. : λ/5.0

Result of wavefront correction at 12.6mA

uncorrected image σ_x=430μm
σ_y=240μm



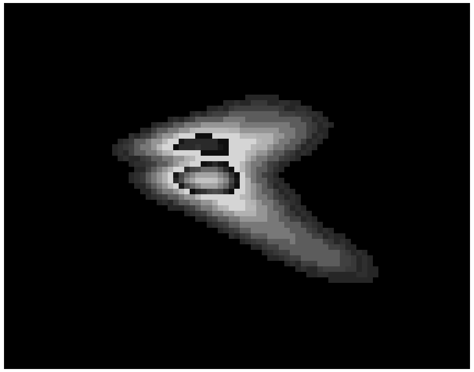
corrected image σ_x=255μm
σ_y=139μm



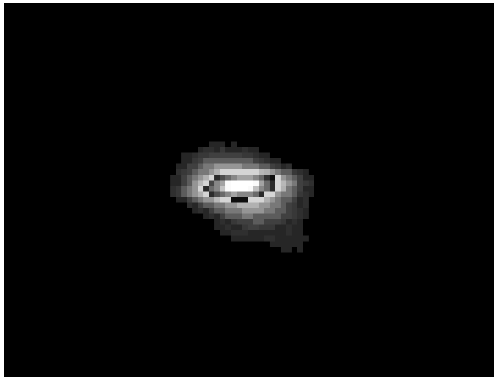
remaining wavefront error p-v : λ/2.7,
rms. : λ/12.5

Result of wavefront correction at 92.0mA

uncorrected image σ_x=250μm σ_y=237μm



corrected image σ_x=254μm σ_y=139μm

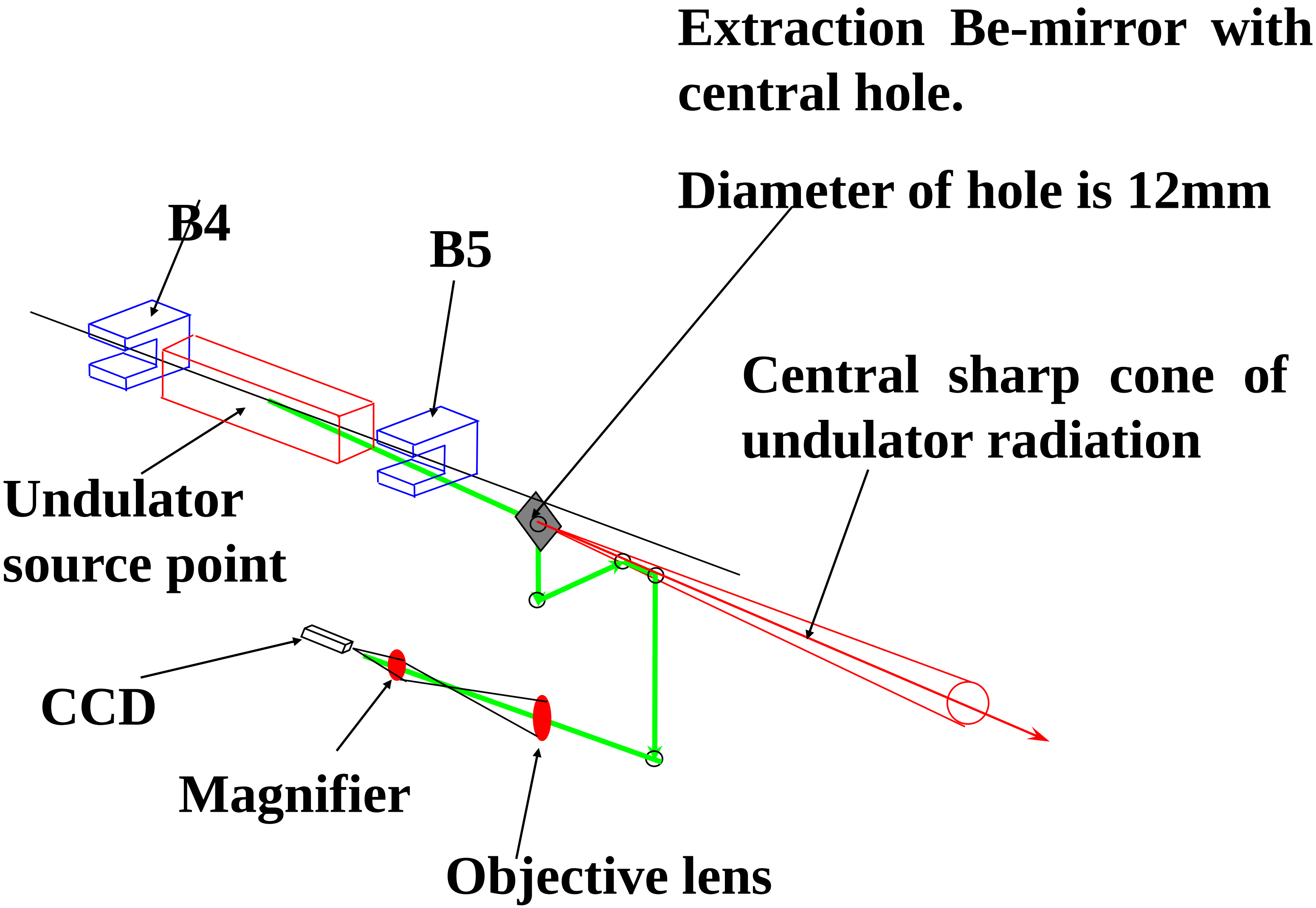


remaining wavefront error p-v : λ/2.7,
rms. : λ/5.0

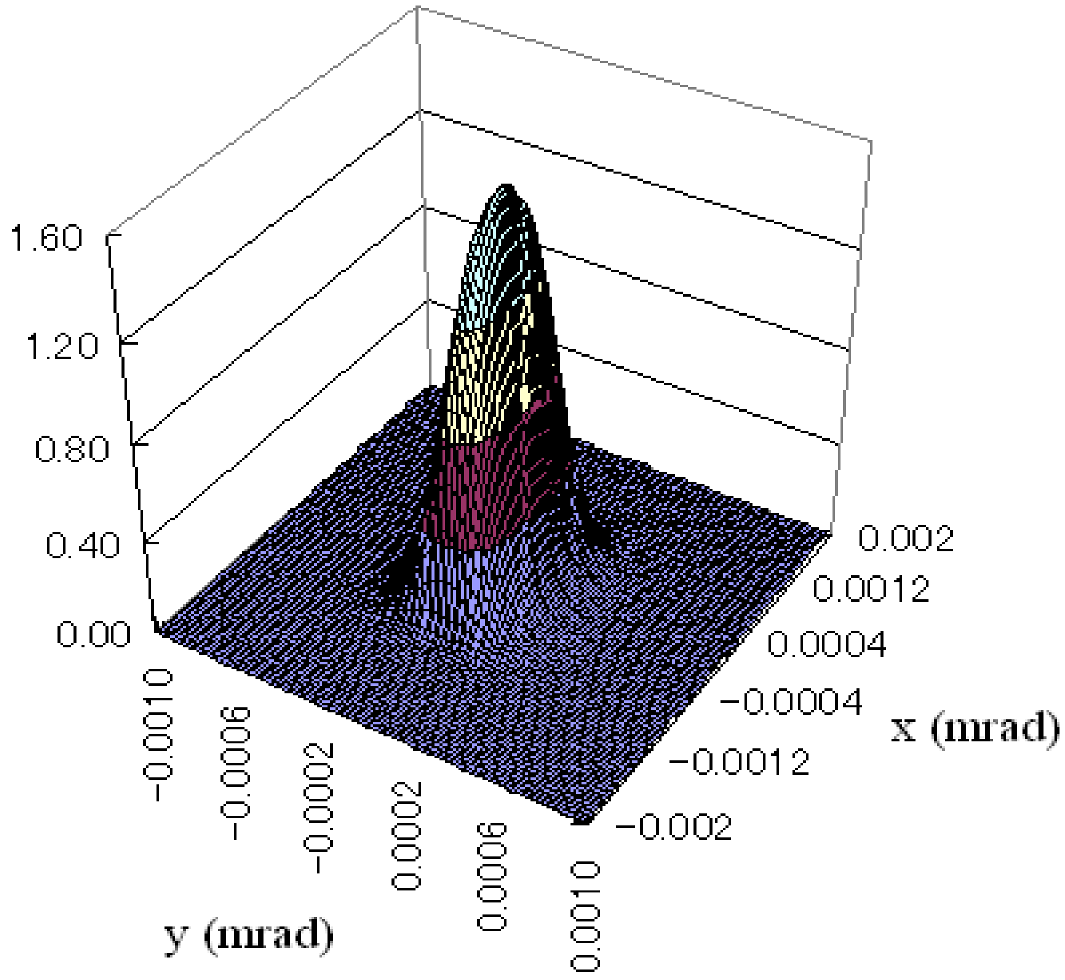
Optical phase space monitor

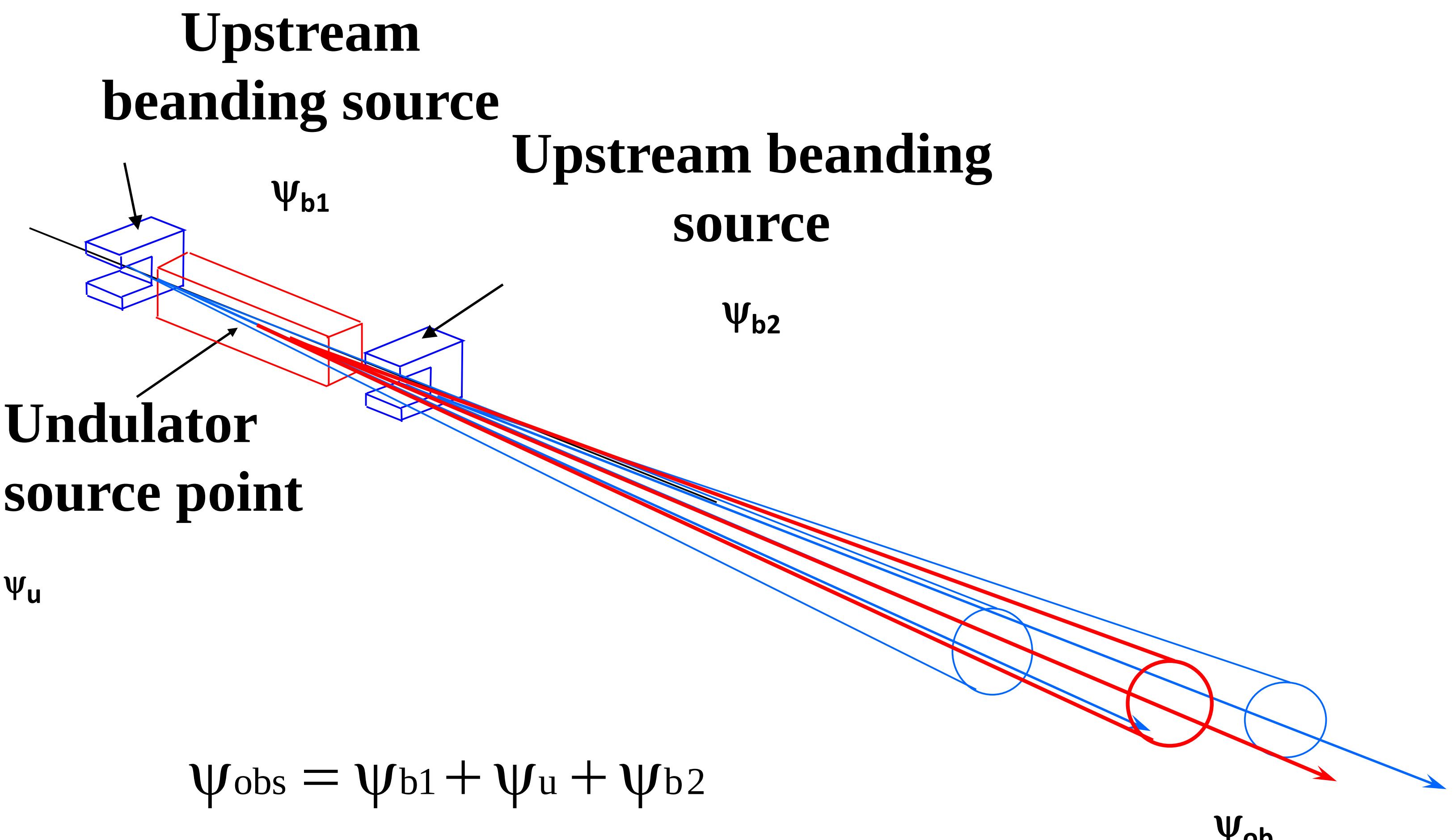
(position and angle of the electron beam)

by focul system and afocal system


Extraction Be-mirror with central hole.
Diameter of hole is 12mm
B4
B5
Central sharp cone of undulator radiation
Undulator source point
CCD
Magnifier
Objective lens

Power density of BL5 Undulation



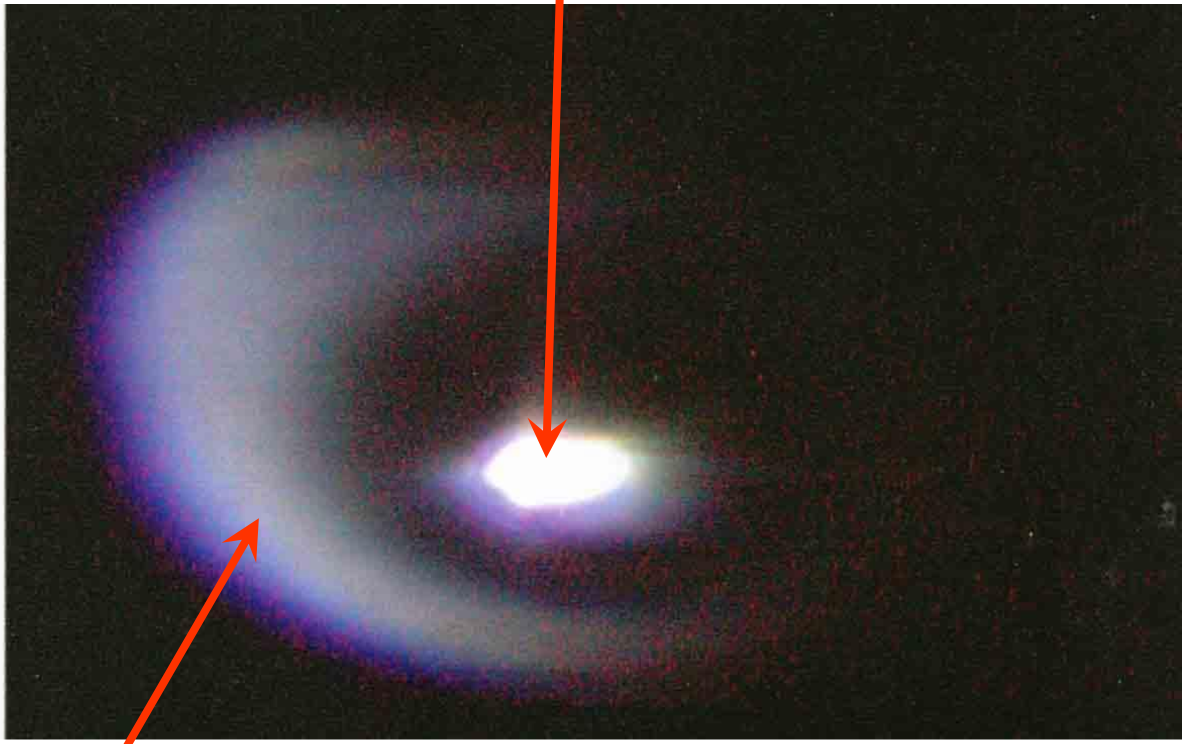


$$\psi_{obs} = \psi_{b1} + \psi_u + \psi_{b2}$$

$$\psi^2_{obs} = \psi^2_{b1} + \psi^2_u + \psi^2_{b2}$$

Focused onto the undulator

Radiation from undulator



Radiation from downstream bend

image of electron beam in downstream bend

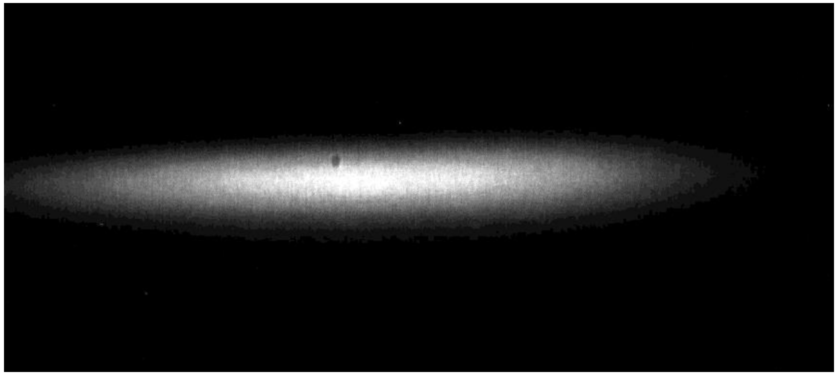
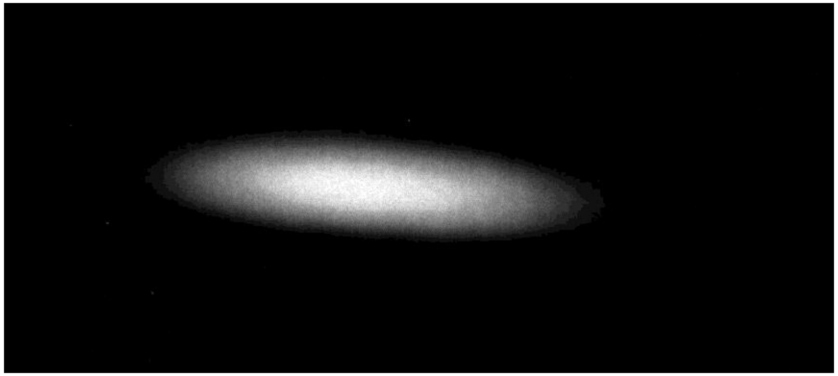


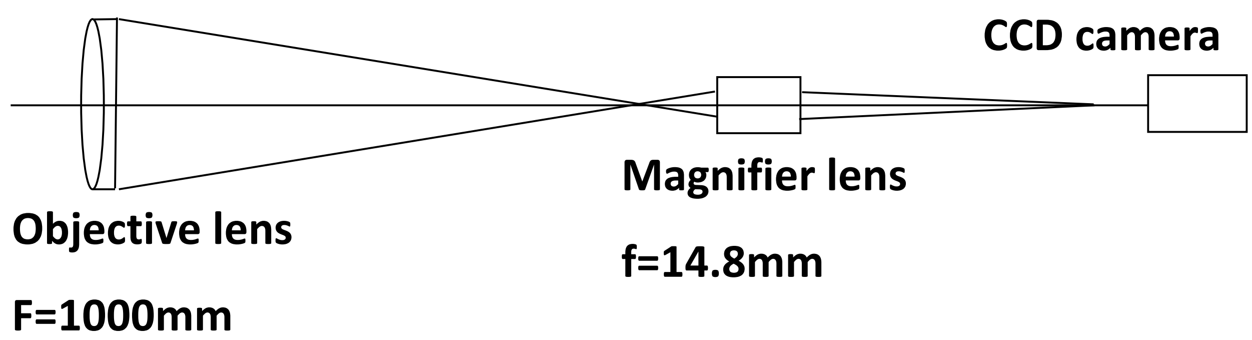
image of electron beam in undulator



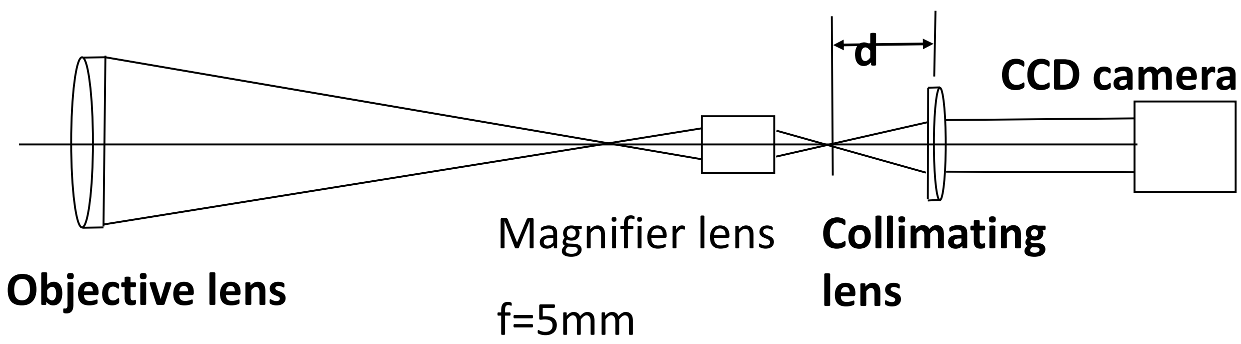
x $\psi_{obs} = \psi_{b1} + \psi_{u} + \psi_{b2}$

o $\psi^2{}_{obs} = \psi^2{}_{b1} + \psi^2{}_{u} + \psi^2{}_{b2}$

Focusing system (a) for the observation of the beam position and afocal system (b) for the observation of beam angle.

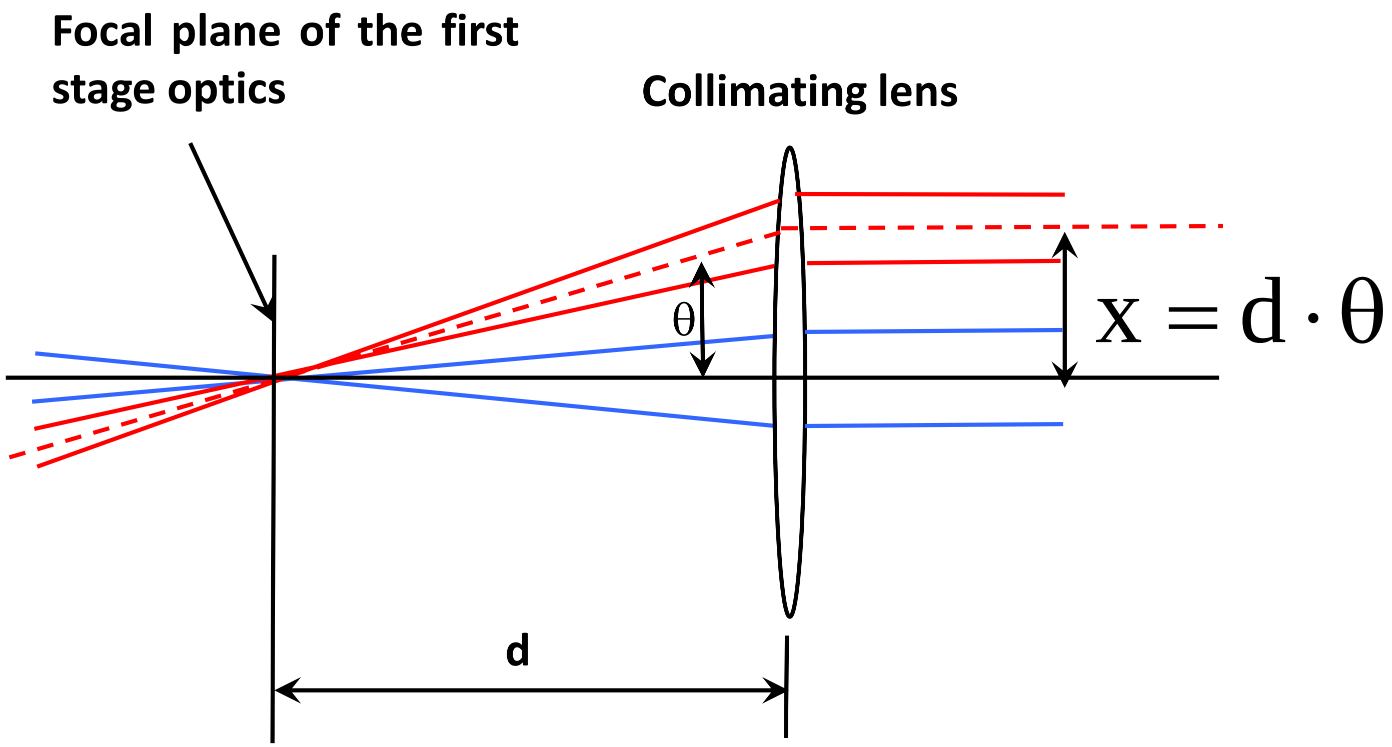


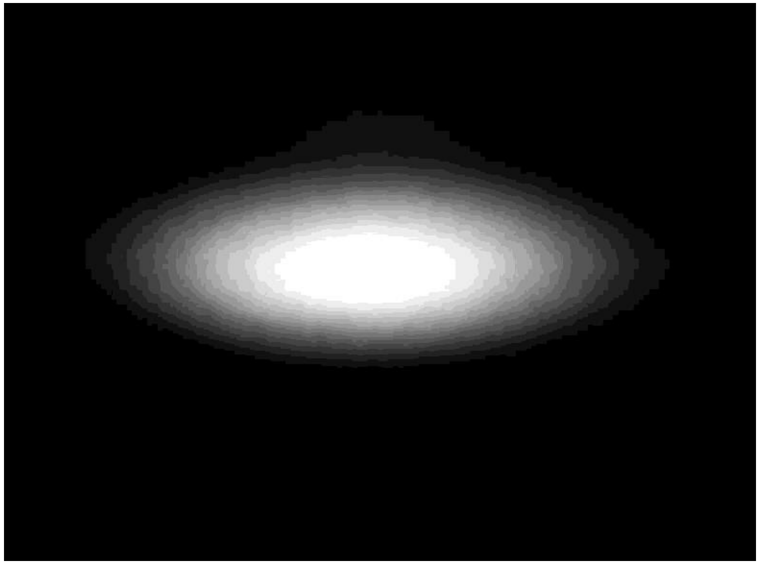
(a)



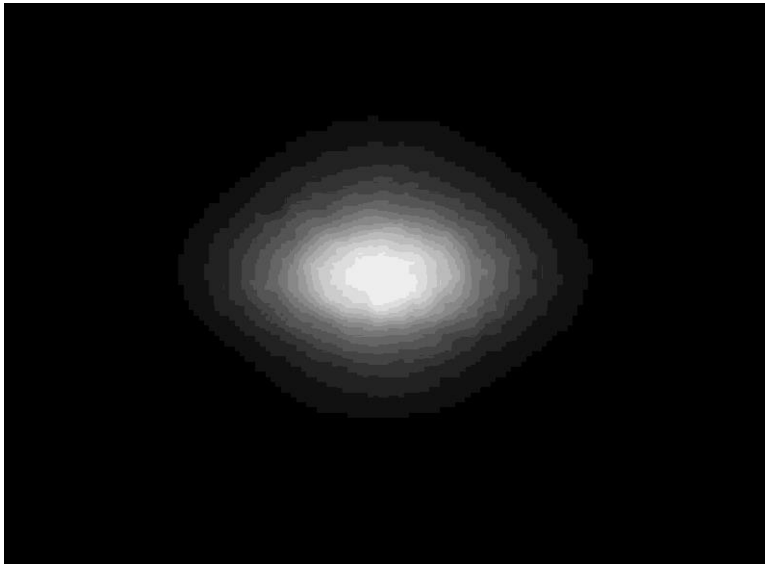
(b)

Collimating section in the afocal system.



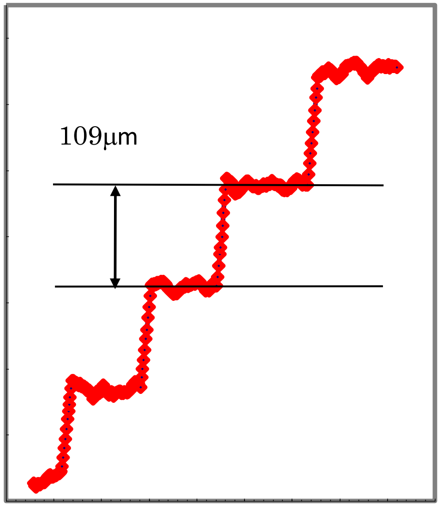


Electron beam image produced by the focusing system.

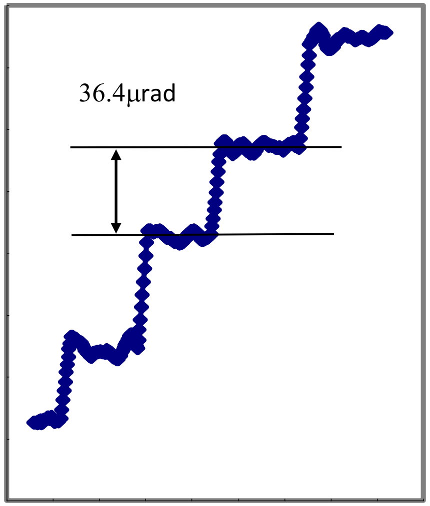


Intensity distribution of the light beam at the afocal system.

Due to a large floor motion at BL5, the calibration and phase space observation were performed at BL21.

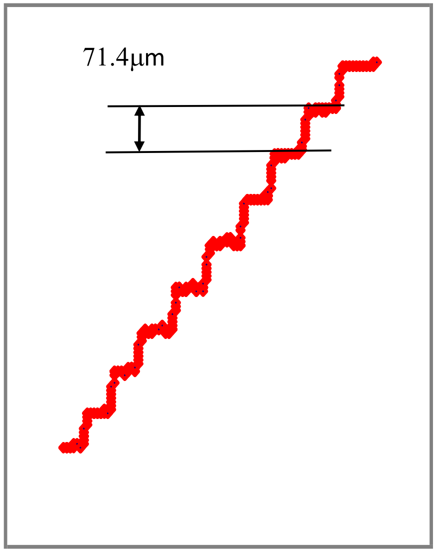


(a) Horizontal position

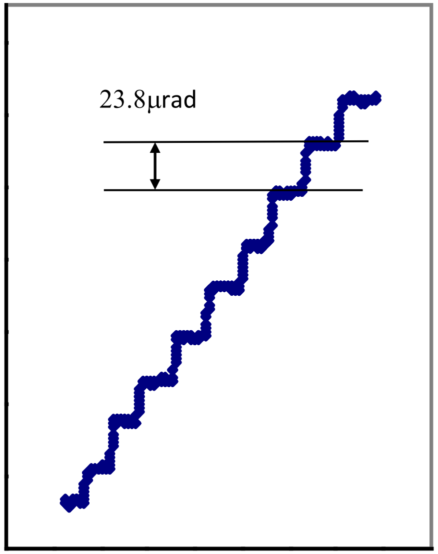


(b) Horizontal angle

Results of calibrations

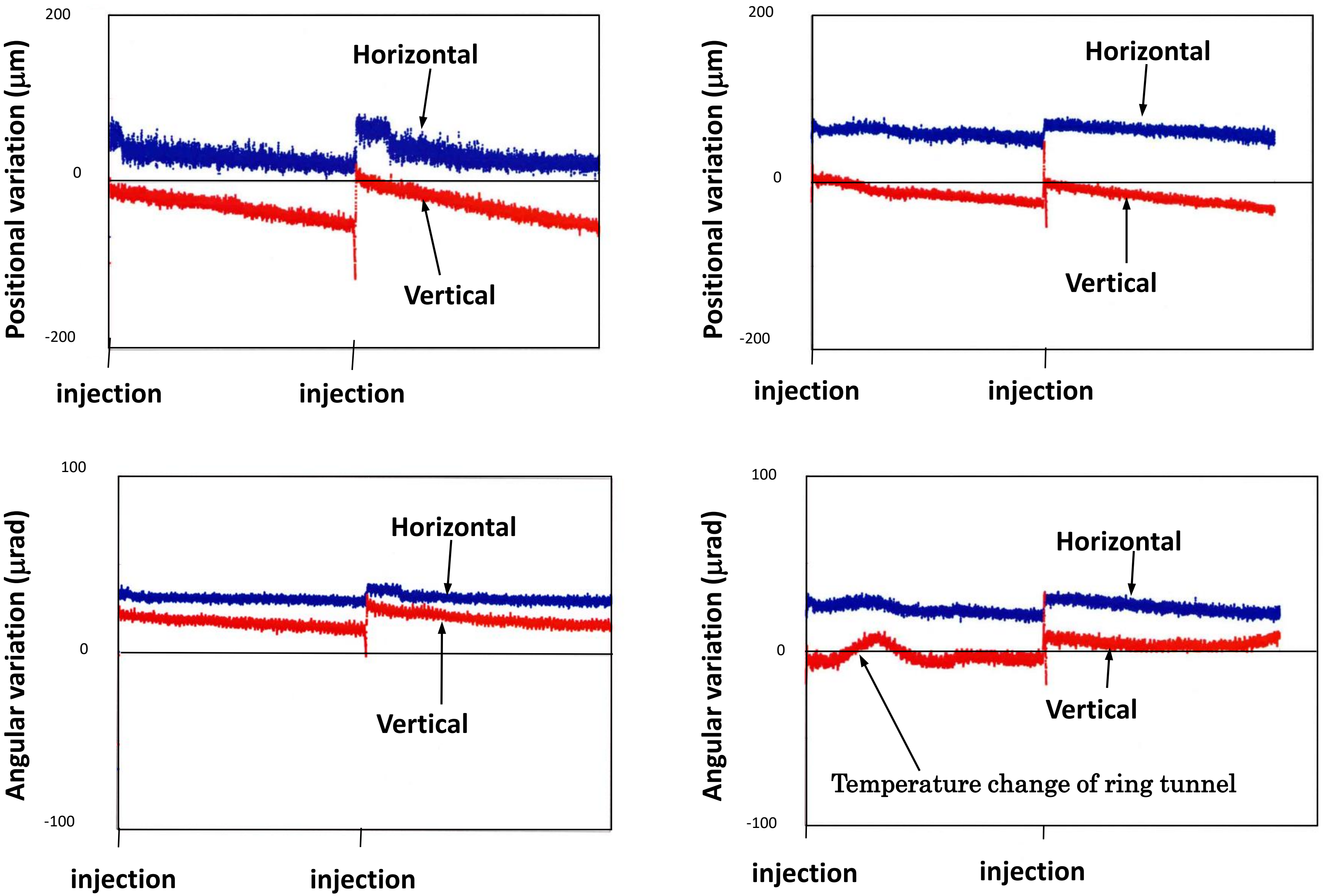


(c) Vertical position

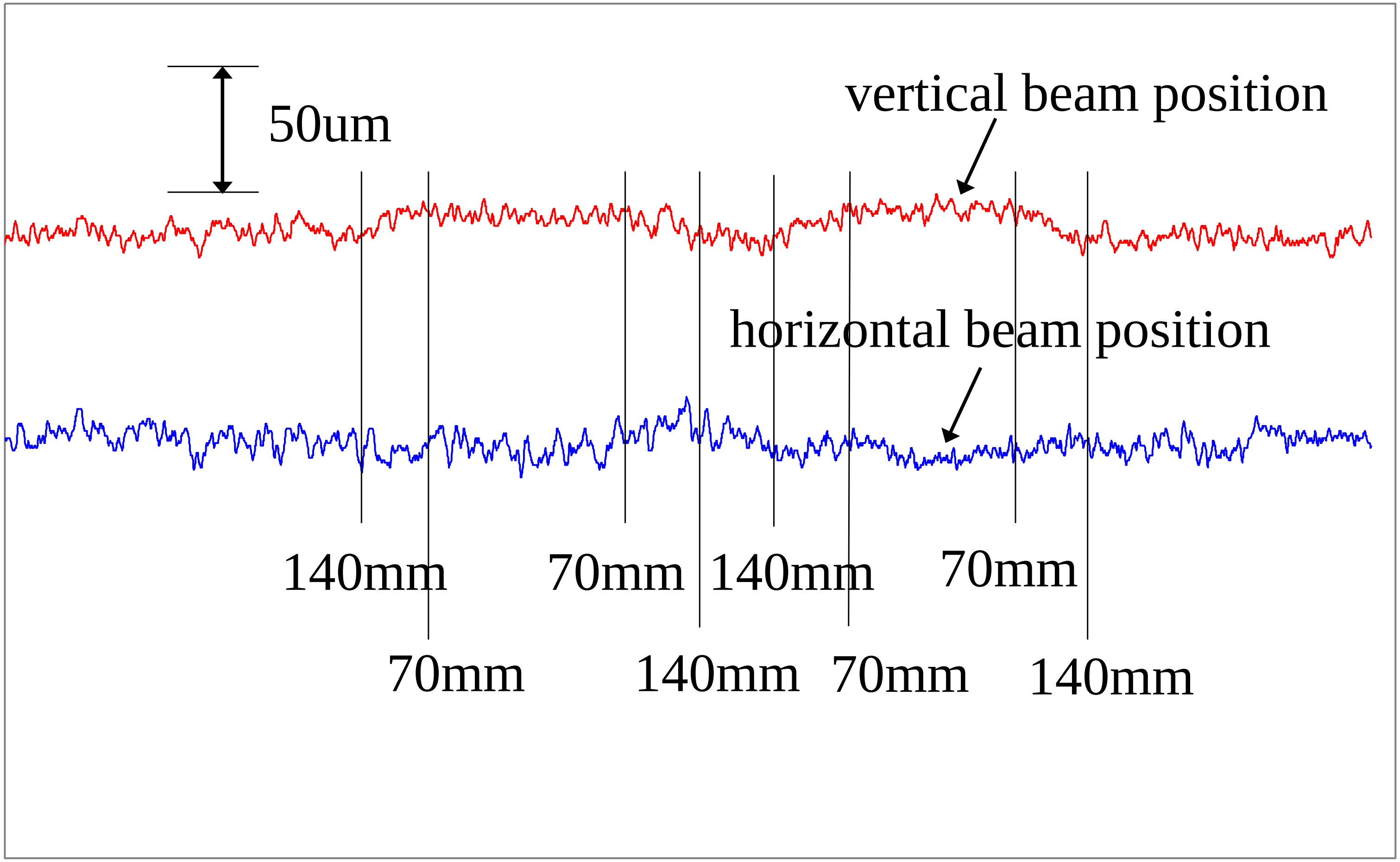


(d) Vertical angle

Result of phase space motion of electron beam at BL21



A result of measurements beam positions during moving the gap from 140mm to 70mm atBL5.



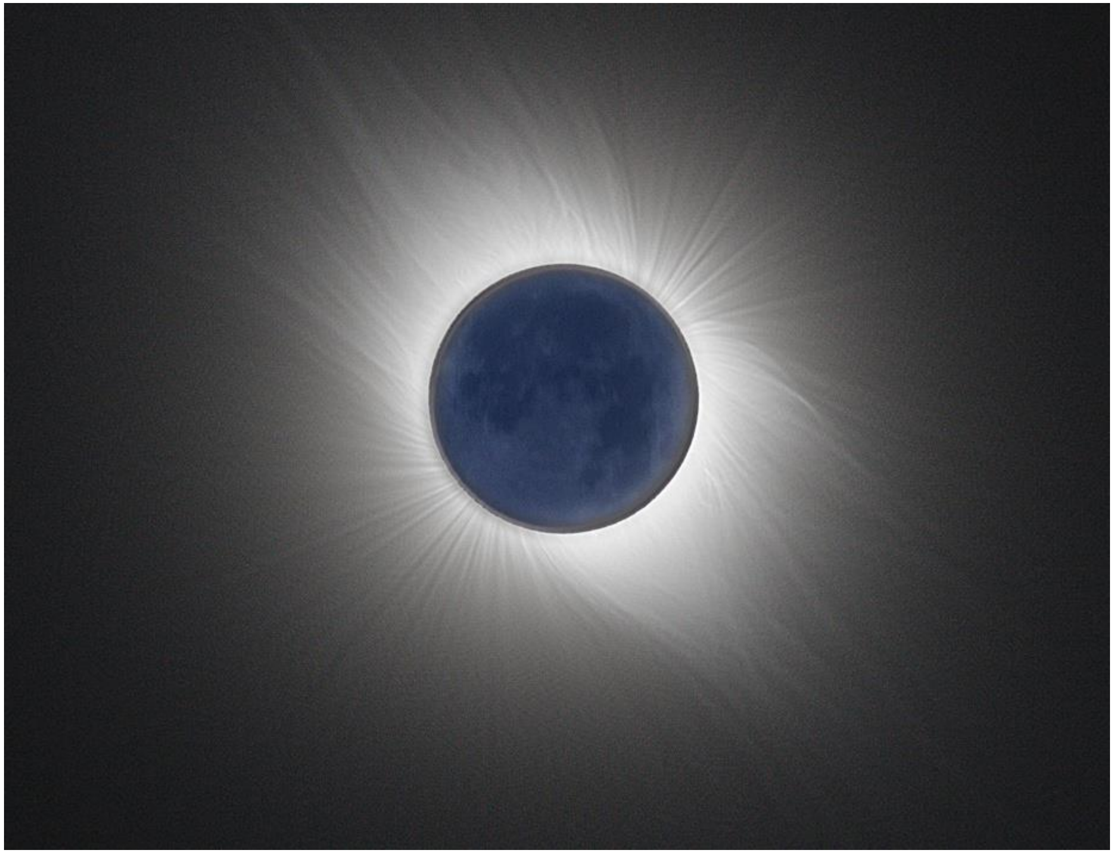
Beam halo measurement with the Coronagraph

The coronagraph to observe sun corona

Developed by B.F.Lyot in 1934 for a observation of sun corona by artificial eclipse.

Special telescope having a
re-diffraction system to eliminate a diffraction fringe.

Everything was start with astronomer's dream……



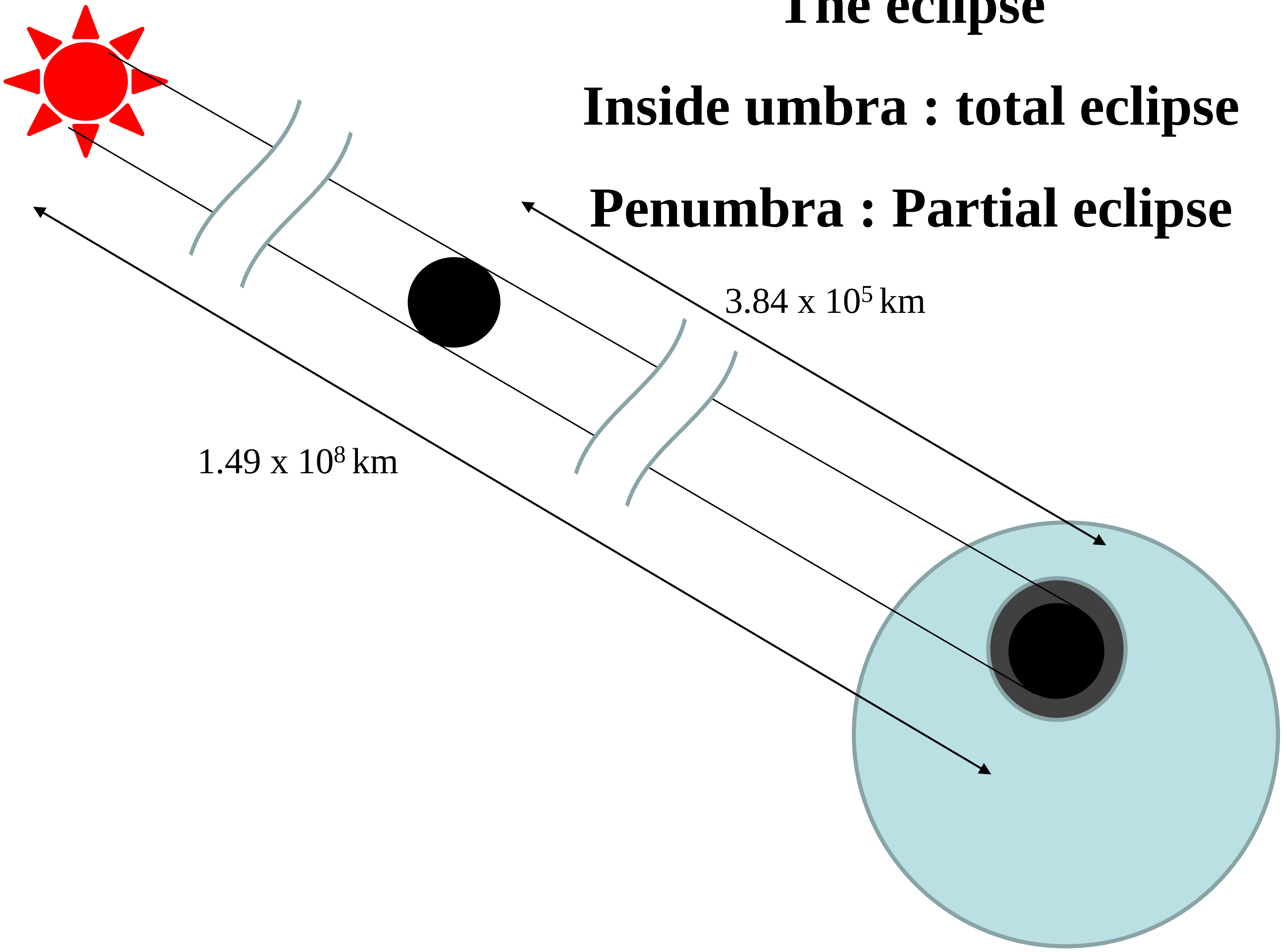
Eclipse is rare phenomena, and only few second is available for observation of sun corona, prominence etc.

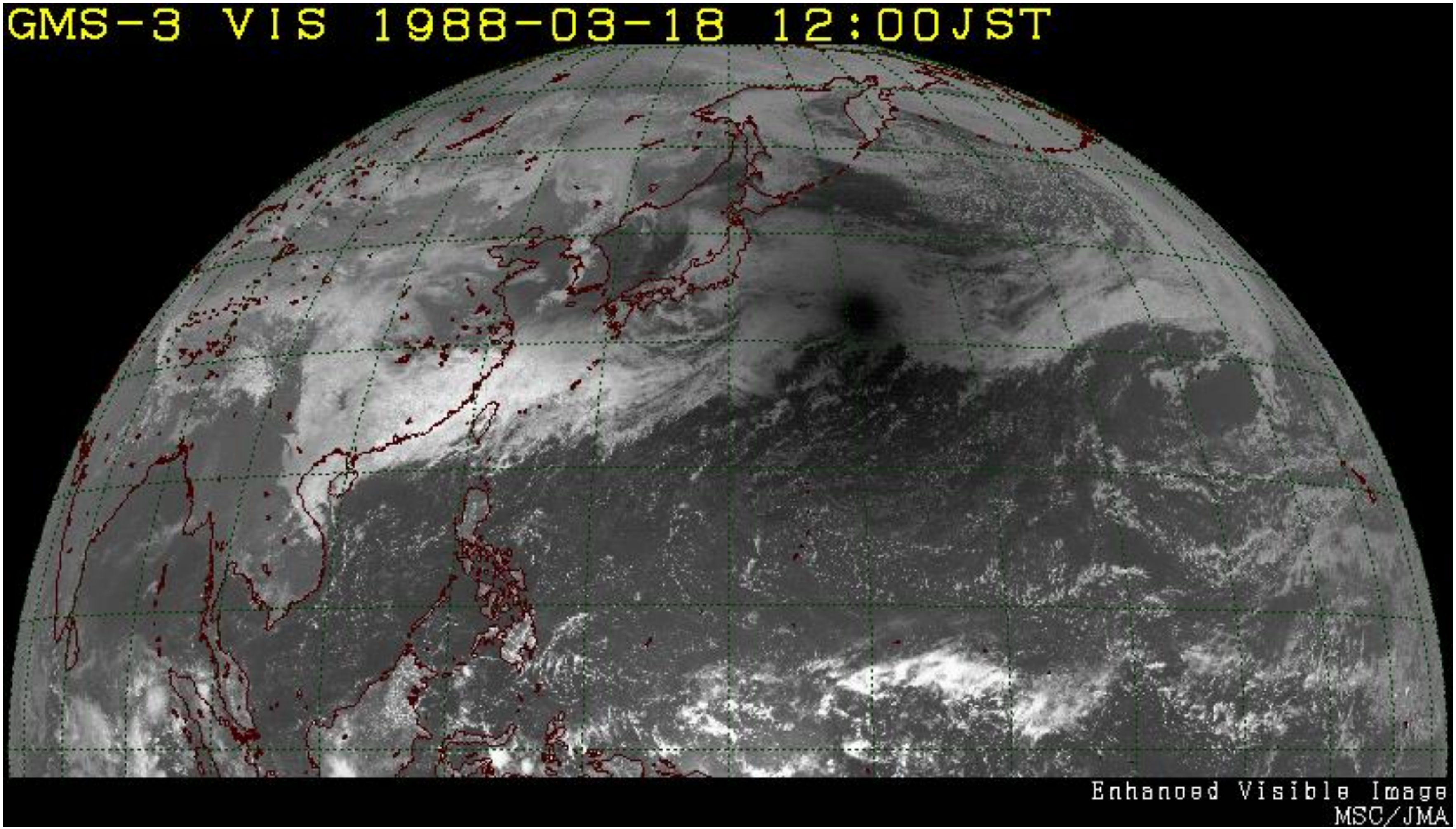
Artificial eclipse was dream of astronomers, but……..

The eclipse

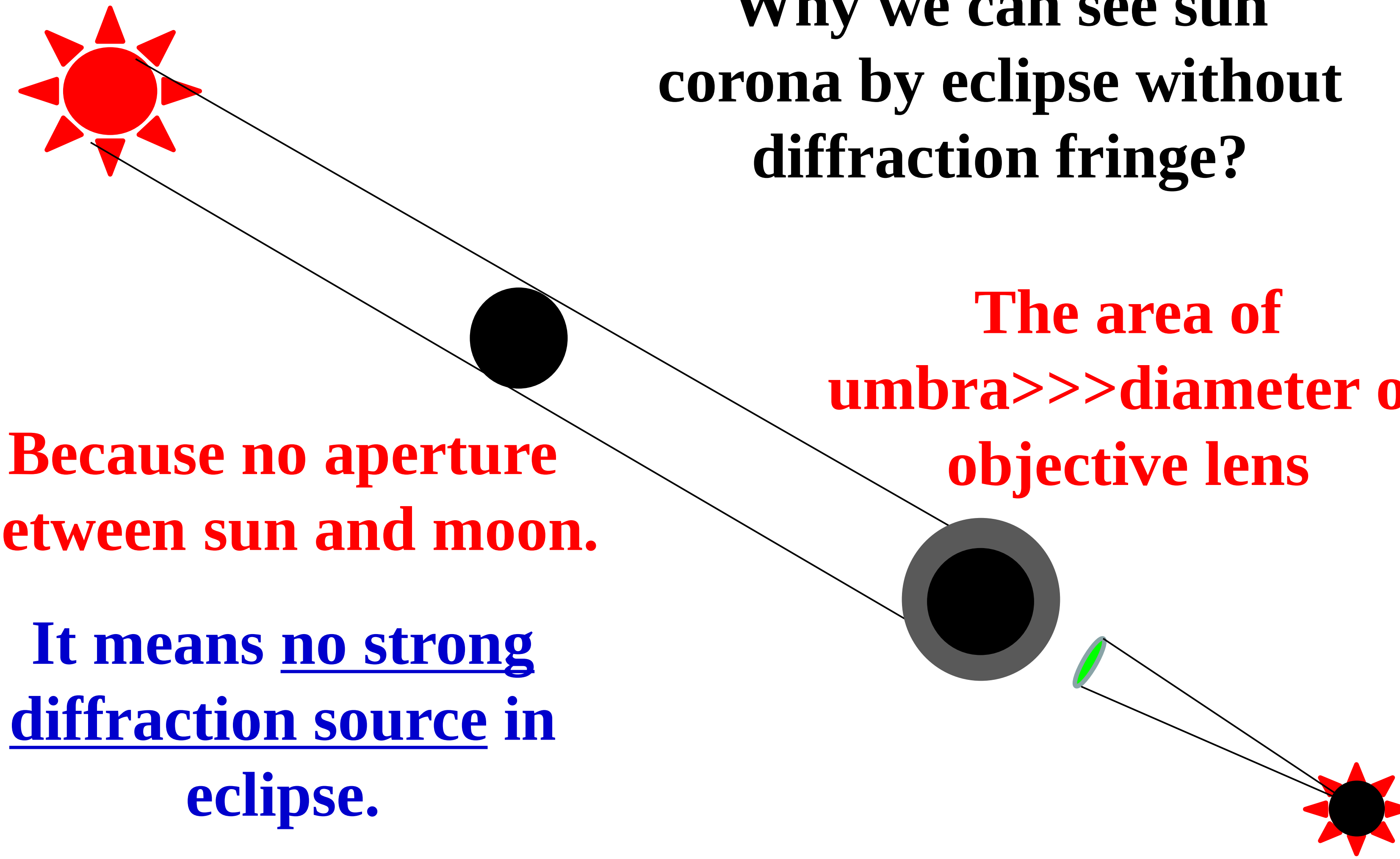
Inside umbra : total eclipse

Penumbra : Partial eclipse




GMS-3 VIS 1988-03-18 12:00JST
Enhanced Visible Image
MSC/JMA

Why we can see sun corona by eclipse without diffraction fringe?



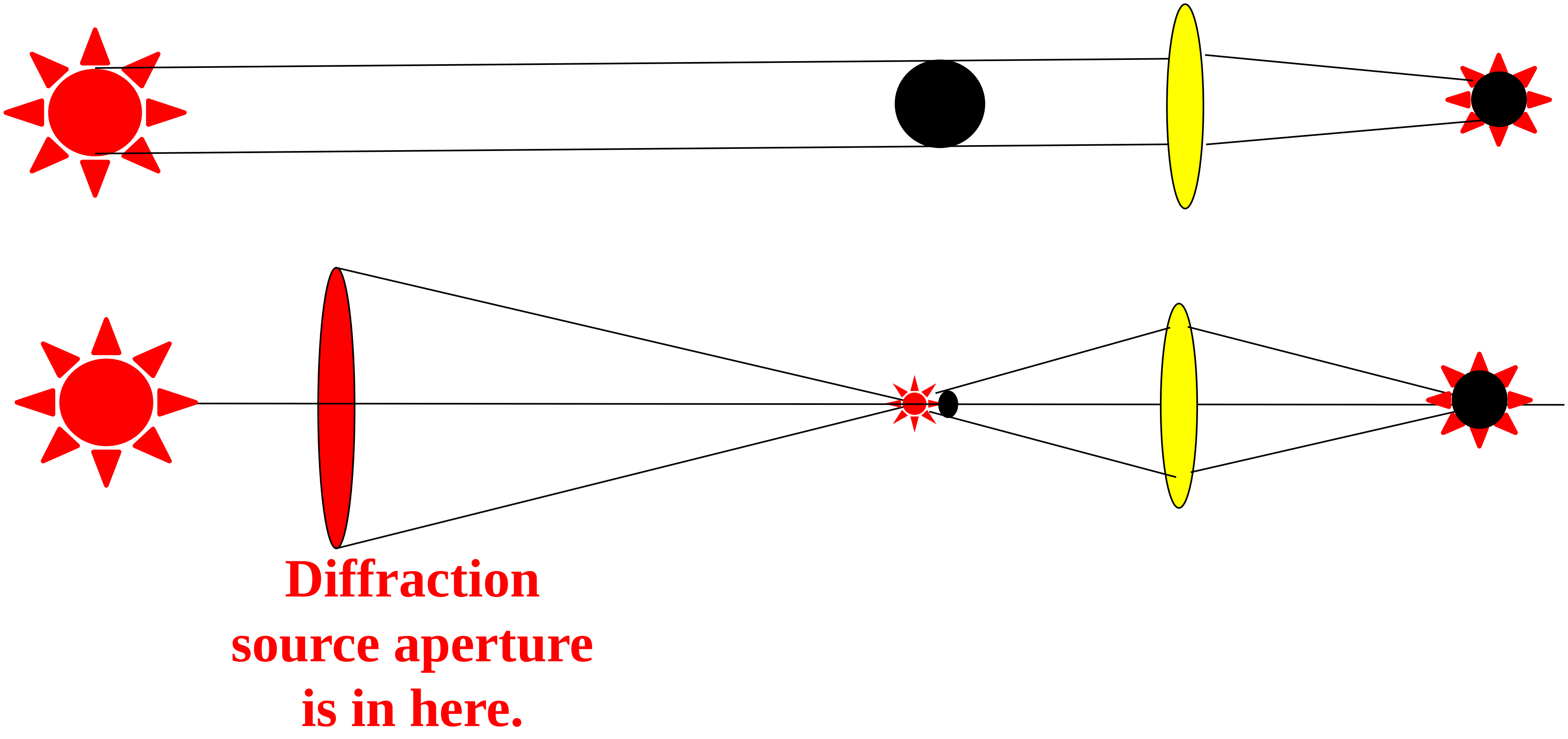
The area of umbra>>>diameter of objective lens

Because no aperture between sun and moon.

It means <u>no strong diffraction source</u> in eclipse.

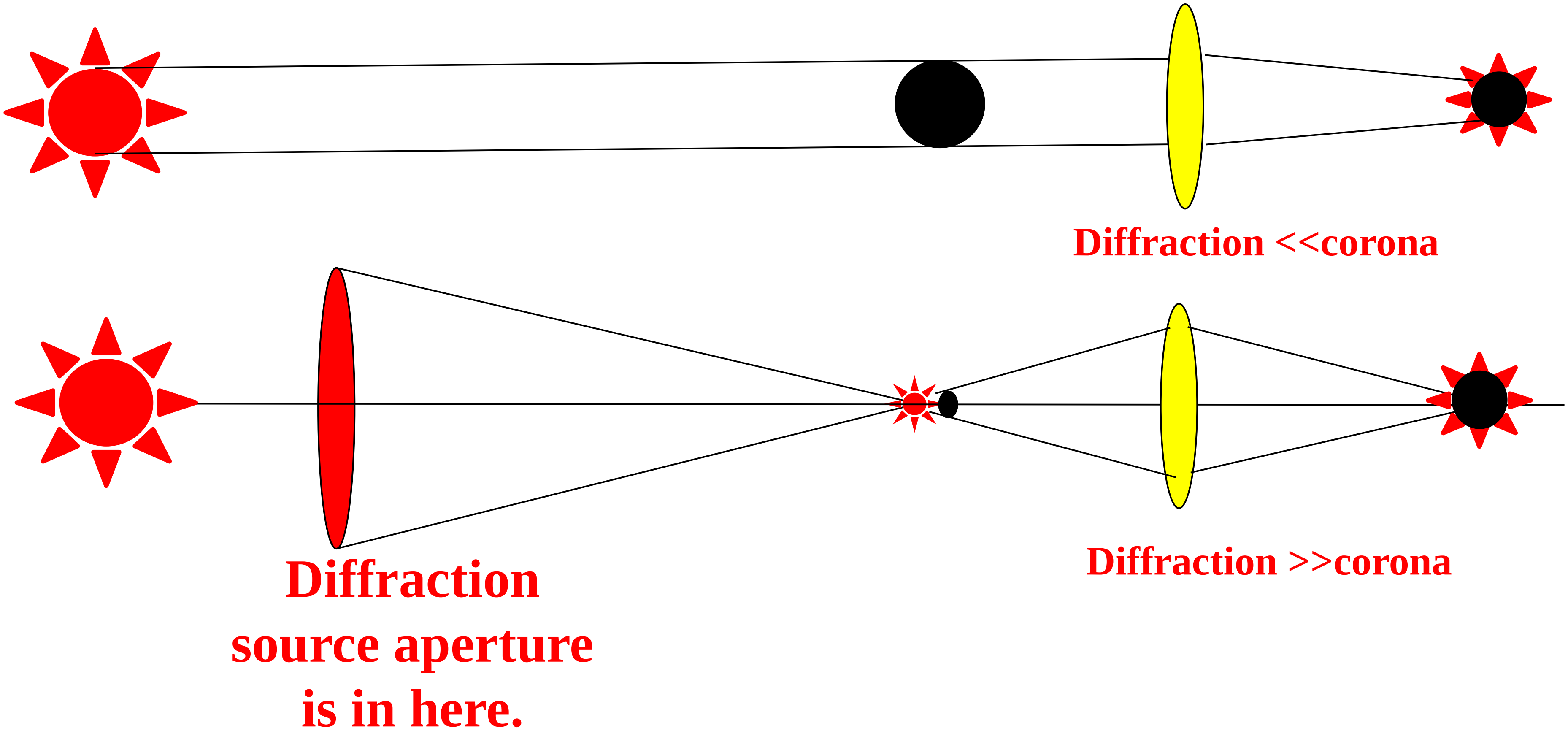
Question is can we make same system with artificial way?

Compare two setup, eclipse and artificial eclipse.



Answer is to eliminate diffraction fringe from aperture. …but how????

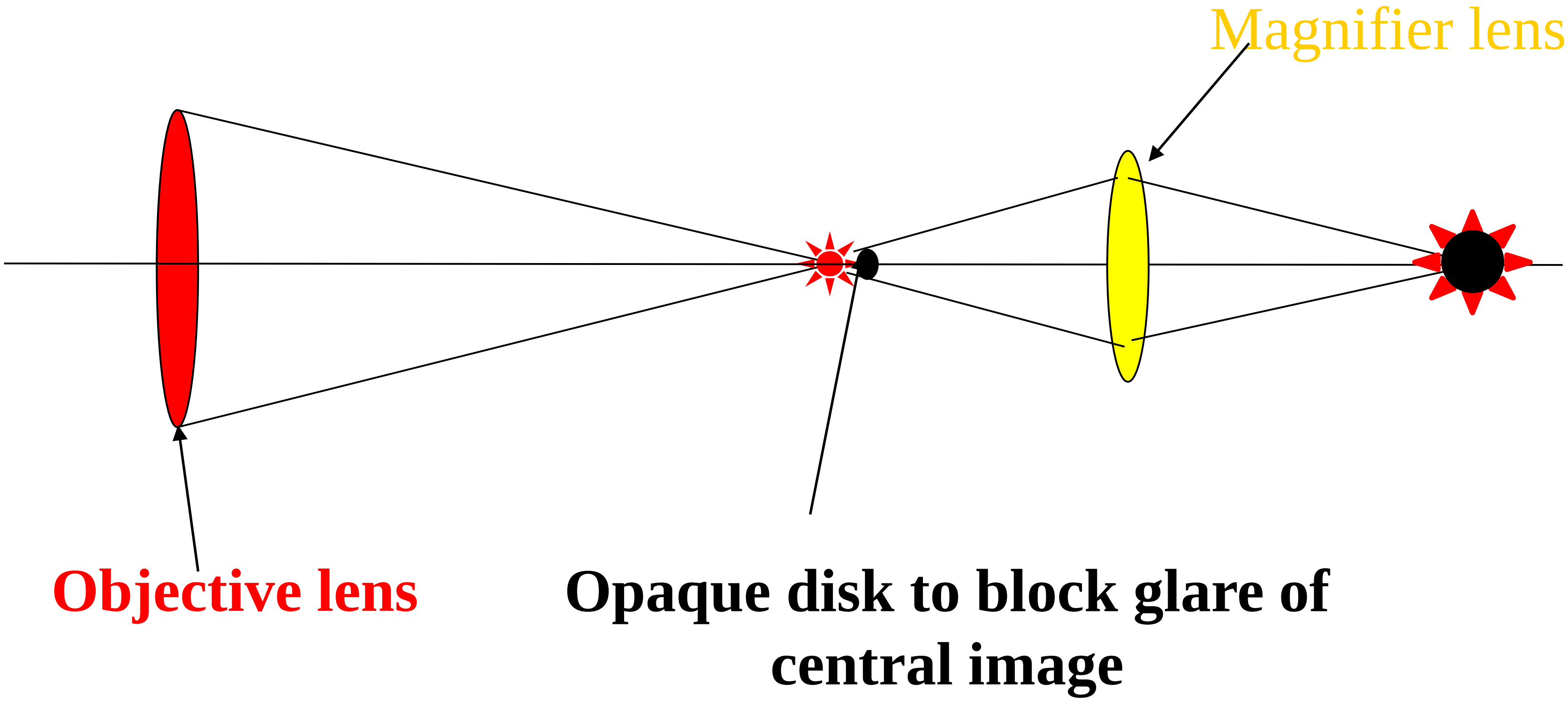
Compare two setup, eclipse and artificial eclipse.

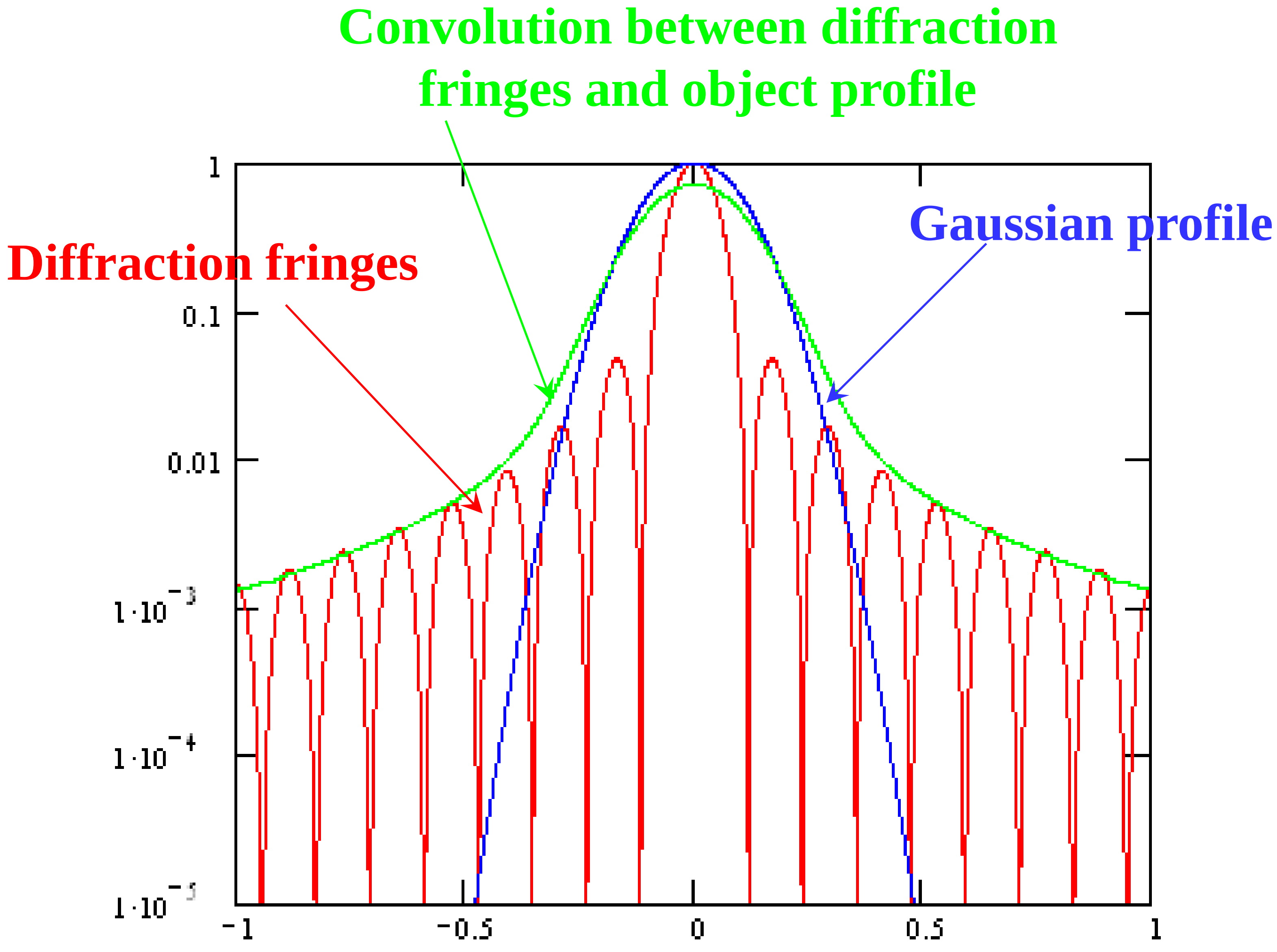


Answer is to eliminate diffraction fringe from aperture. …but how????

1. Diffraction fringes vs. halo

Observation with normal telescope

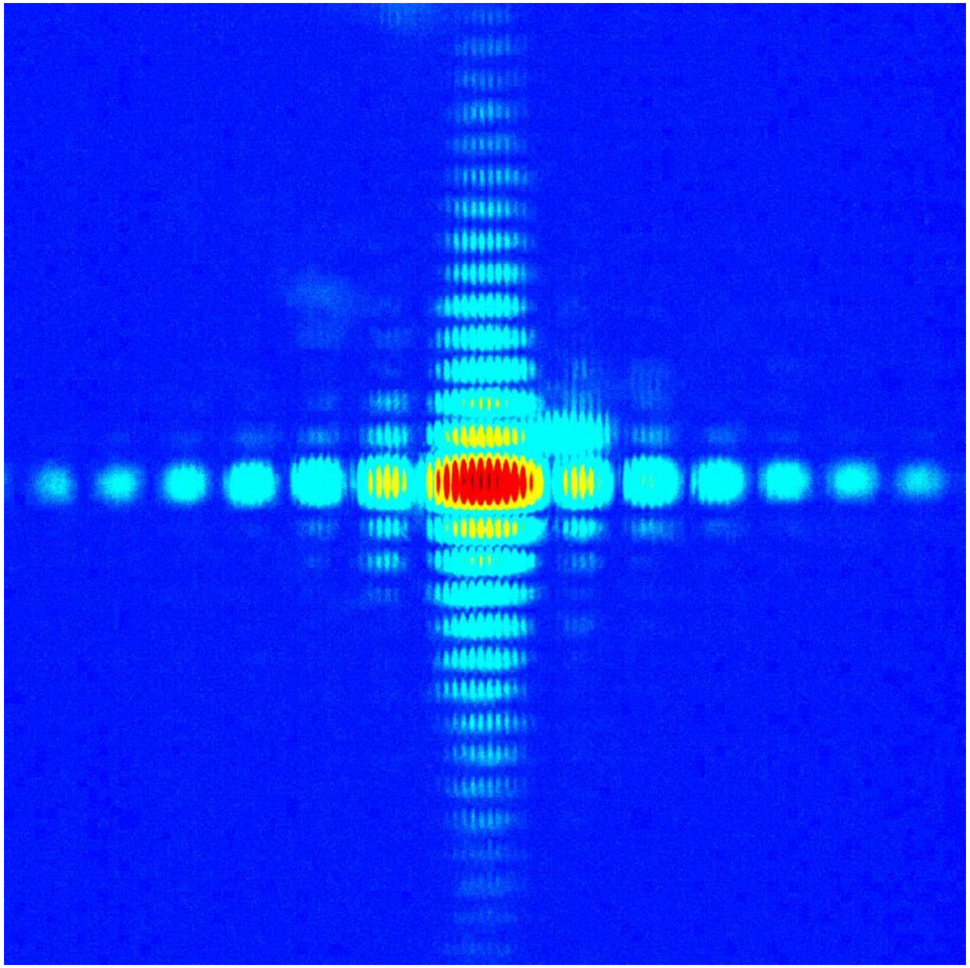




Convolution between diffraction
fringes and object profile
Gaussian profile
Diffraction fringes
1
0.1
0.01
1·10^-3
1·10^-4
1·10^-5
-1
-0.5
0
0.5
1

Why the coronagraph is necessary?

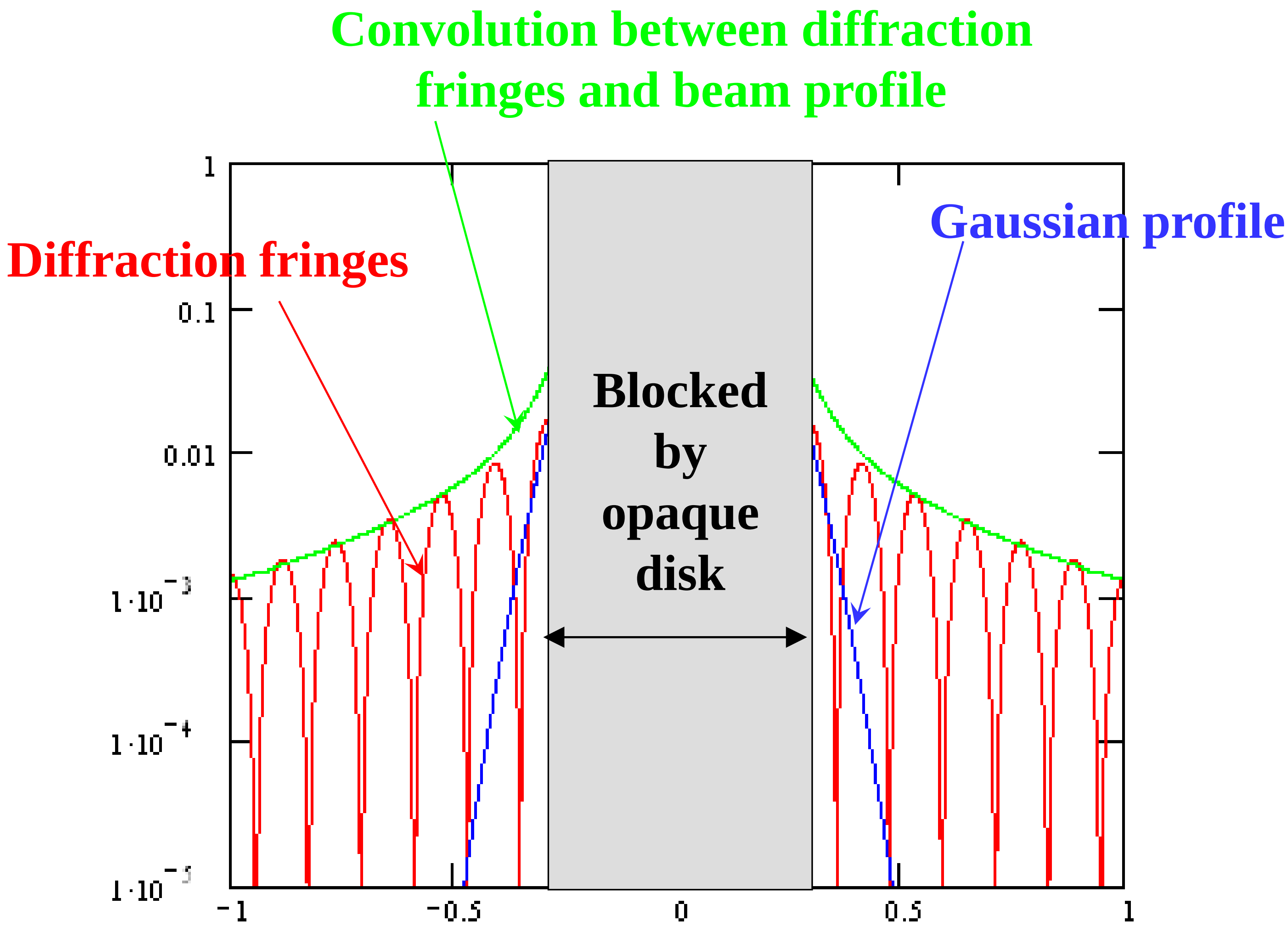
Diffraction pattern (with interference fringe) taken by sCMOS low noise and high dynamic range camera at ASLS



Diffraction makes fringes surrounding from the central beam image.

Intensity of diffraction fringes are in the range of 10^{-2} -10^{-3} of the peak intensity.

The diffraction fringes disturb observation of week object corona surrounding which has intensity range of 10^{-5} -10^{-6} of bright sun sphere.

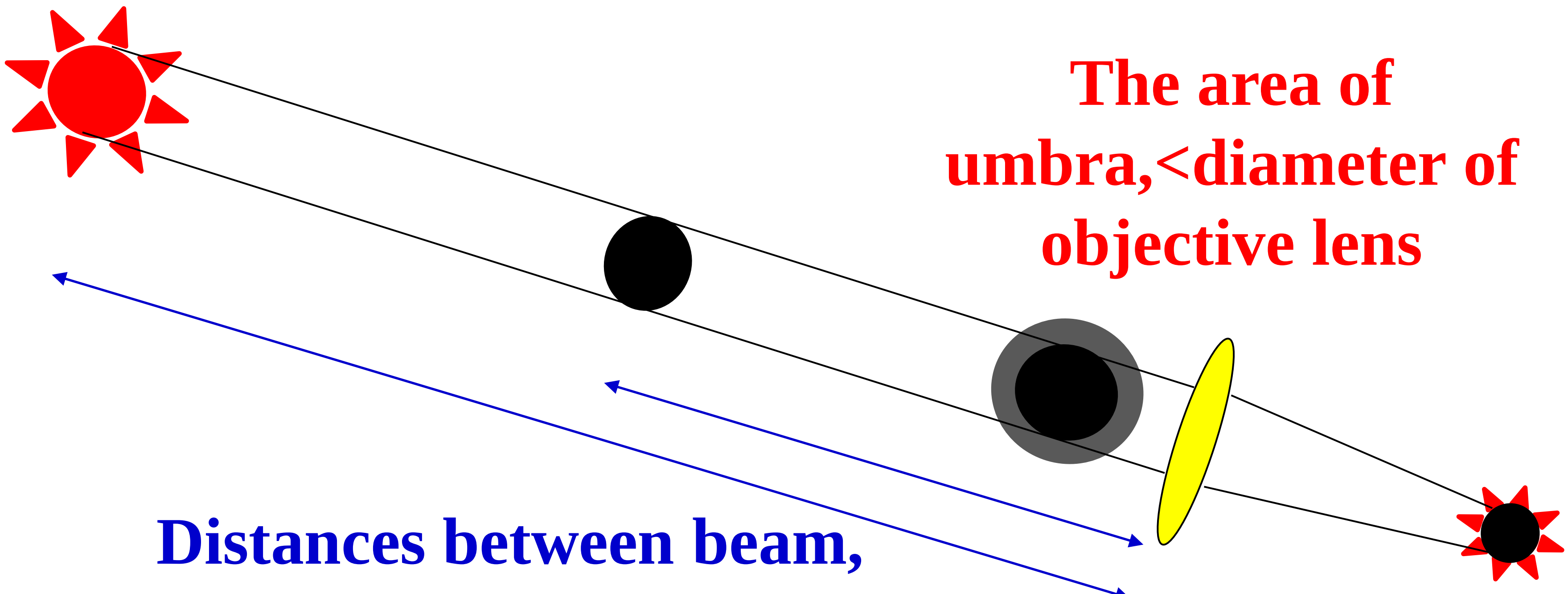

Convolution between diffraction fringes and beam profile
Gaussian profile
Diffraction fringes
Blocked by opaque disk
1
0.1
0.01
$1\cdot10^{-3}$
$1\cdot10^{-4}$
$1\cdot10^{-5}$
−1
−0.5
0
0.5
1

By following reasons, we cannot observe beam halo by using same condition of eclipse

1. Due to small beam size, the umbra size is small, and we cannot put an objective lens with enough diameter.

2. Distances between beam and opaque disk are too short, we cannot focus onto both of them in the same focal plane.

Can we make same system for the observation of beam halo?

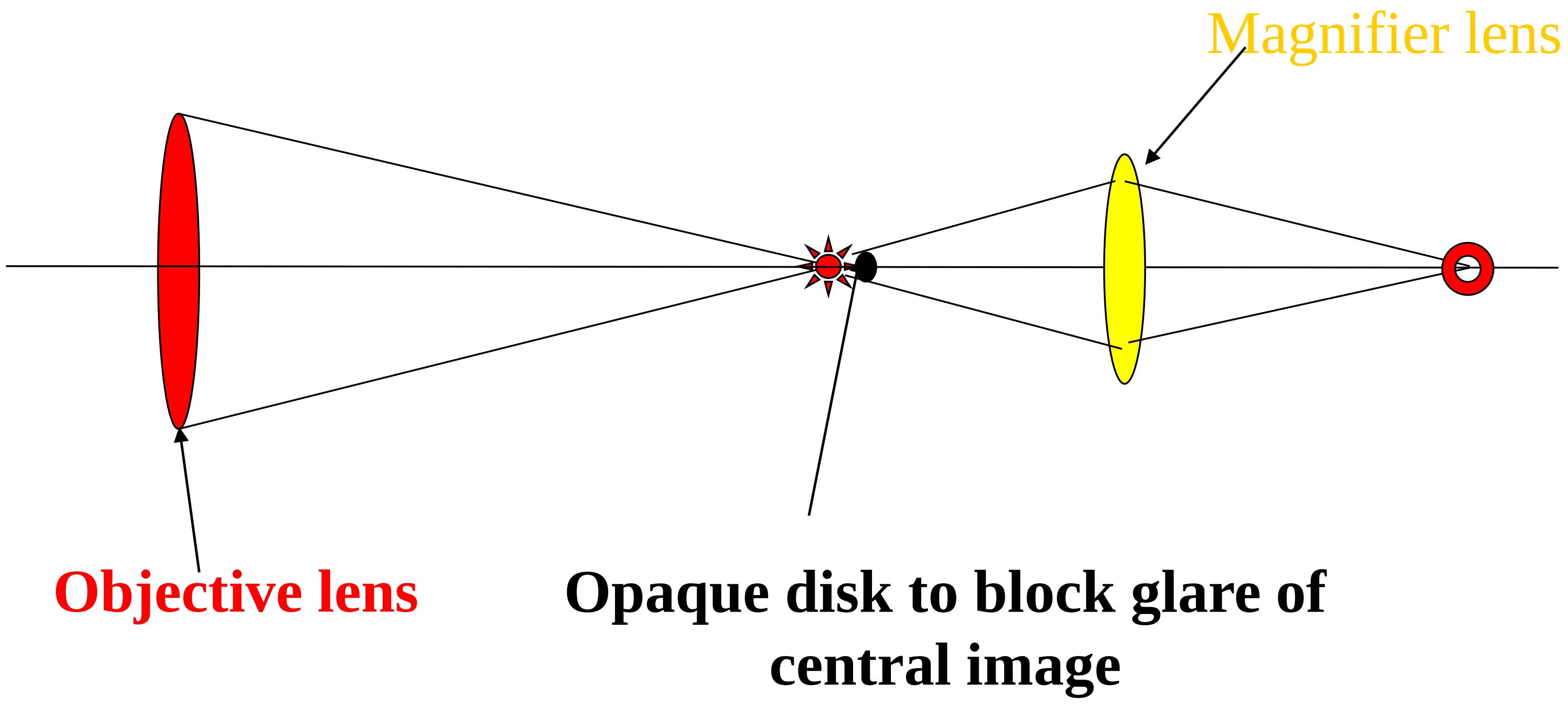


The area of umbra,<diameter of objective lens

Distances between beam, opaque disk and objective lens are short and beam is too small!

Diffraction fringes vs. beam halo

Observation with normal telescope



Diffraction fringes makes tail surrounding from the central beam image.

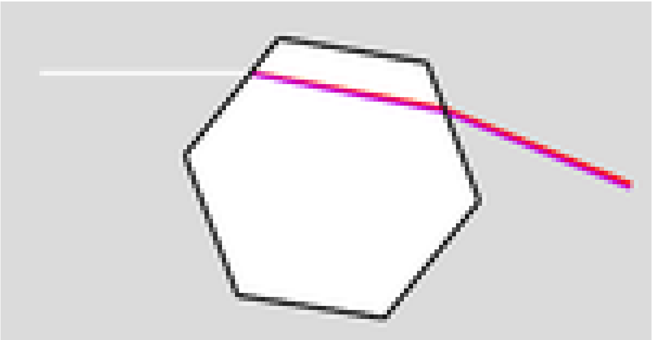
Intensity of diffraction tail is in the range of 10^{-2} -10^{-3} of the peak intensity.

The diffraction tail disturb an observation of week object surrounding from bright central beam

Sun halo

Moon halo

Sun light is refracted by hexagonal ice crystal and making 22deg halo



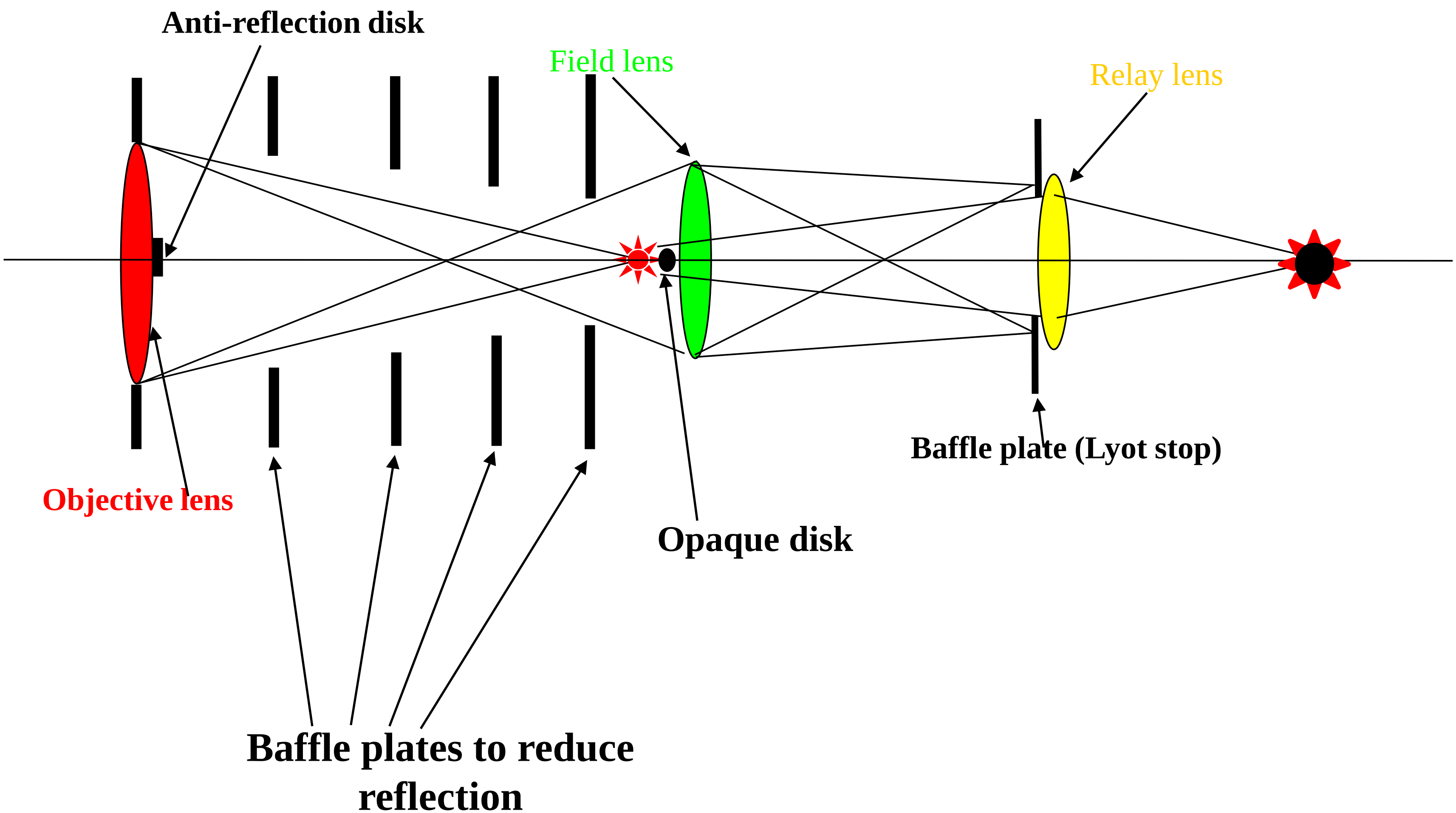
The coronagraph to observe sun corona

Developed by B.F.Lyot in 1934 for a observation of sun corona by artificial eclipse.

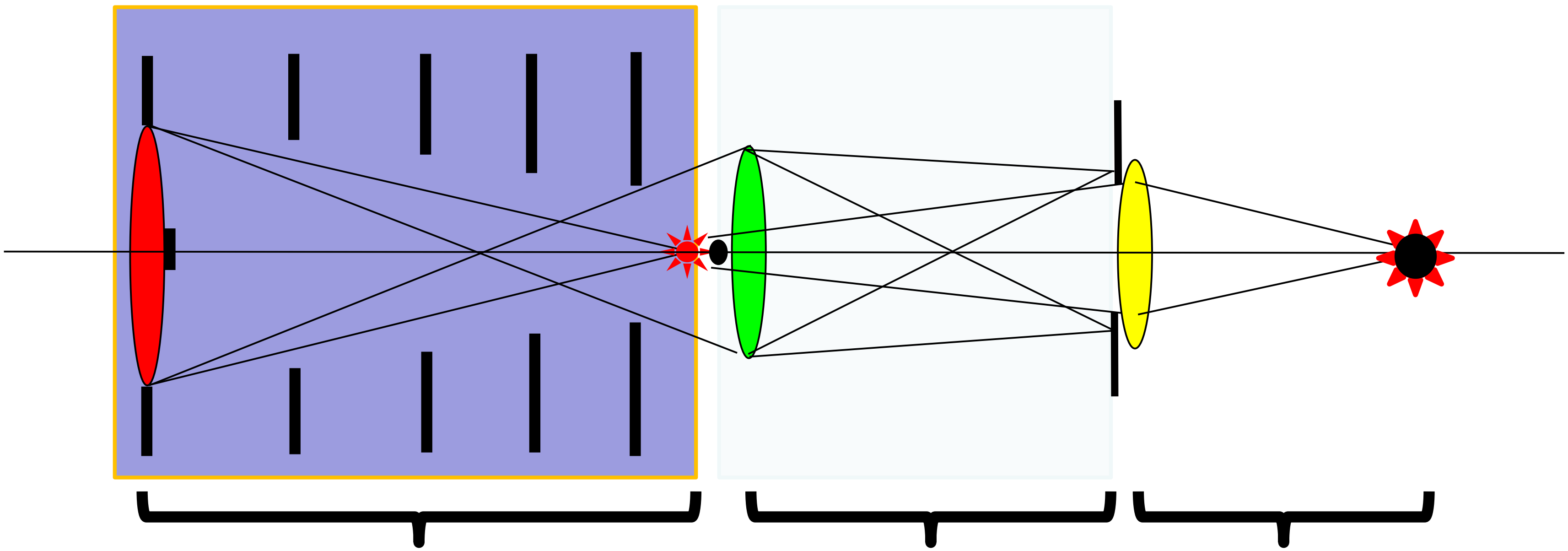
Special telescope having a "*re-diffraction system*" to eliminate the diffraction fringe.

Following explanation for the coronagraph is based on in section 26 special optical system in "Wave optics" written by H.Kubota (unfortunately, this good textbook is written in Japanese).

Optical system of Lyot's corona graph



3 stages-optical system in the Lyot's coronagraph



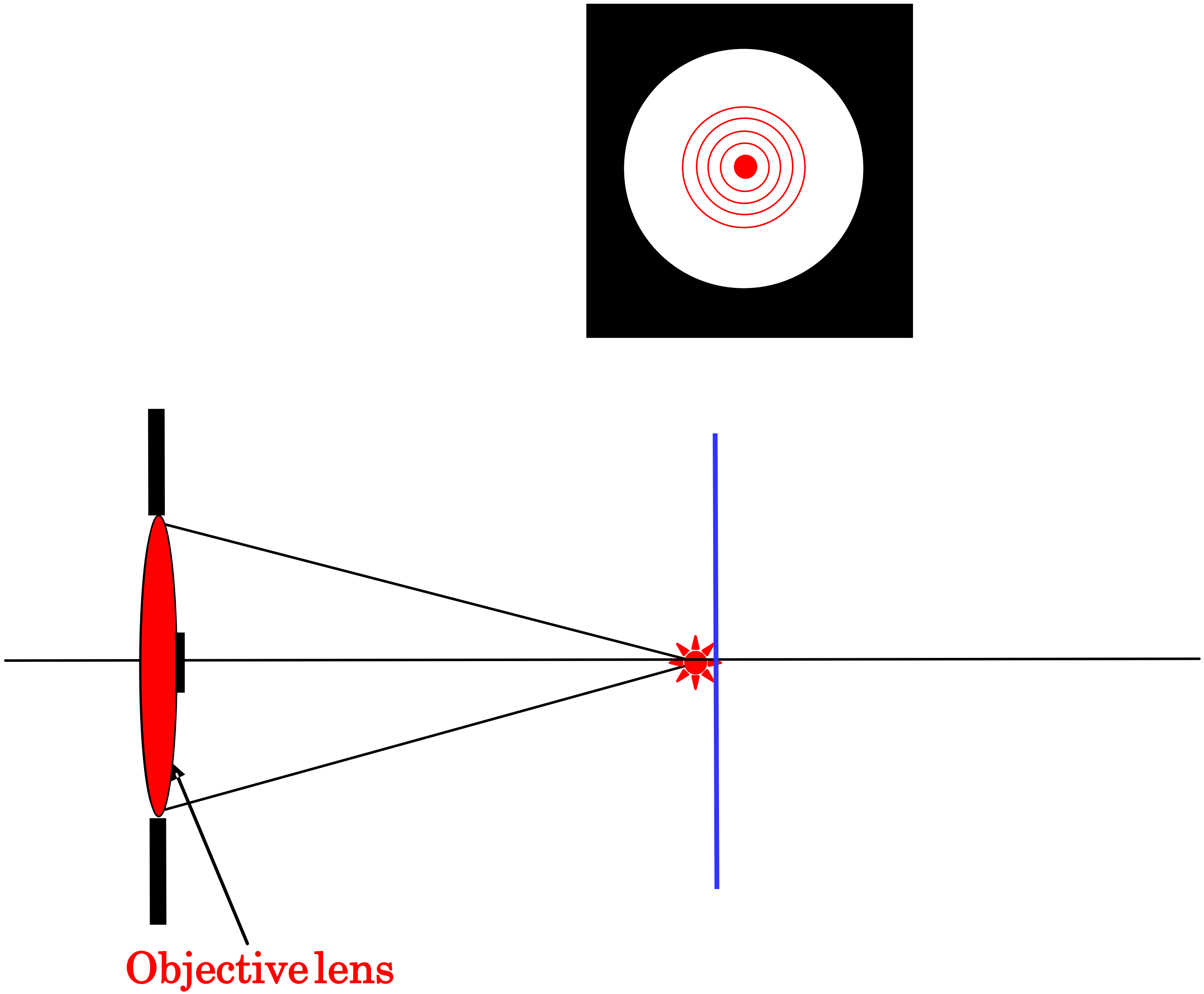
1st stage : Objective lens system

2ed stage : re-diffraction system

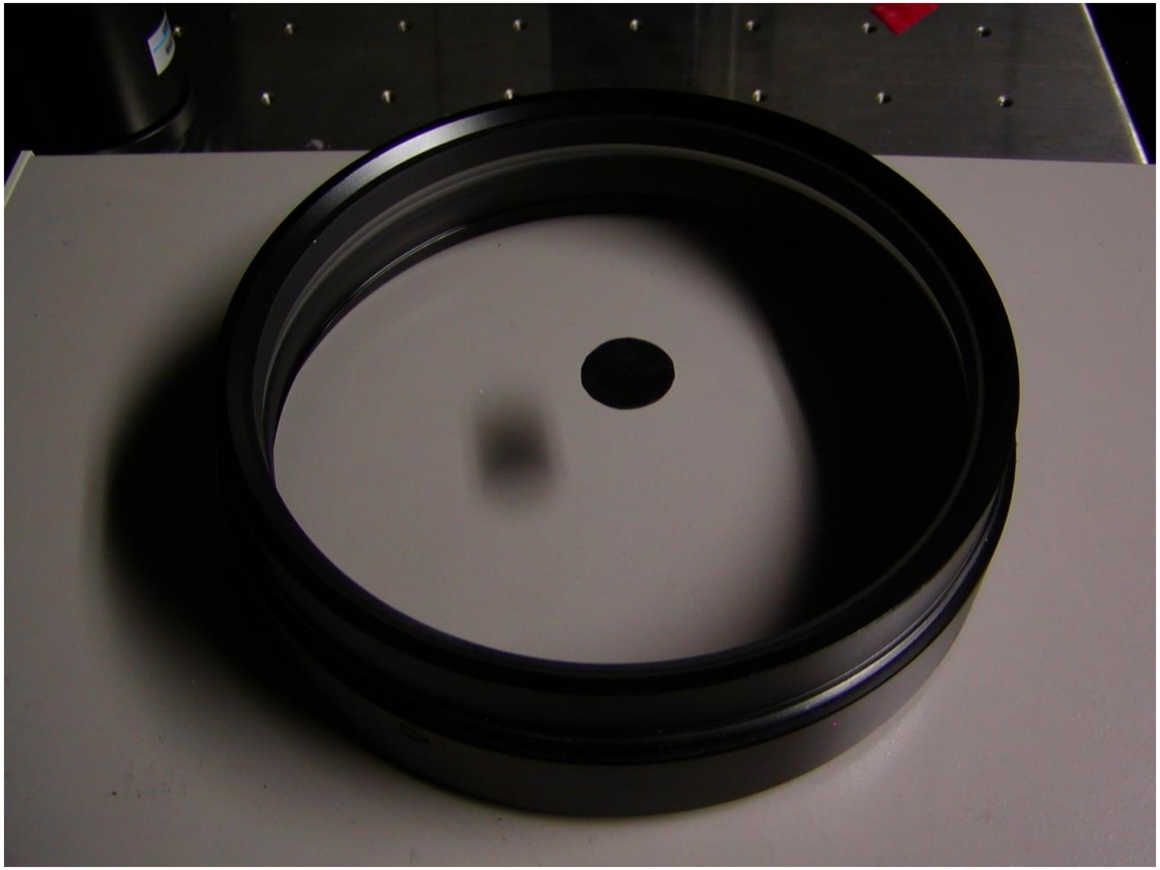
3ed stage :

Relay lens

1. Diffraction of the objective lens


Objective lens

Objective lens with anti-reflection disk to block reflected light from opaque disk



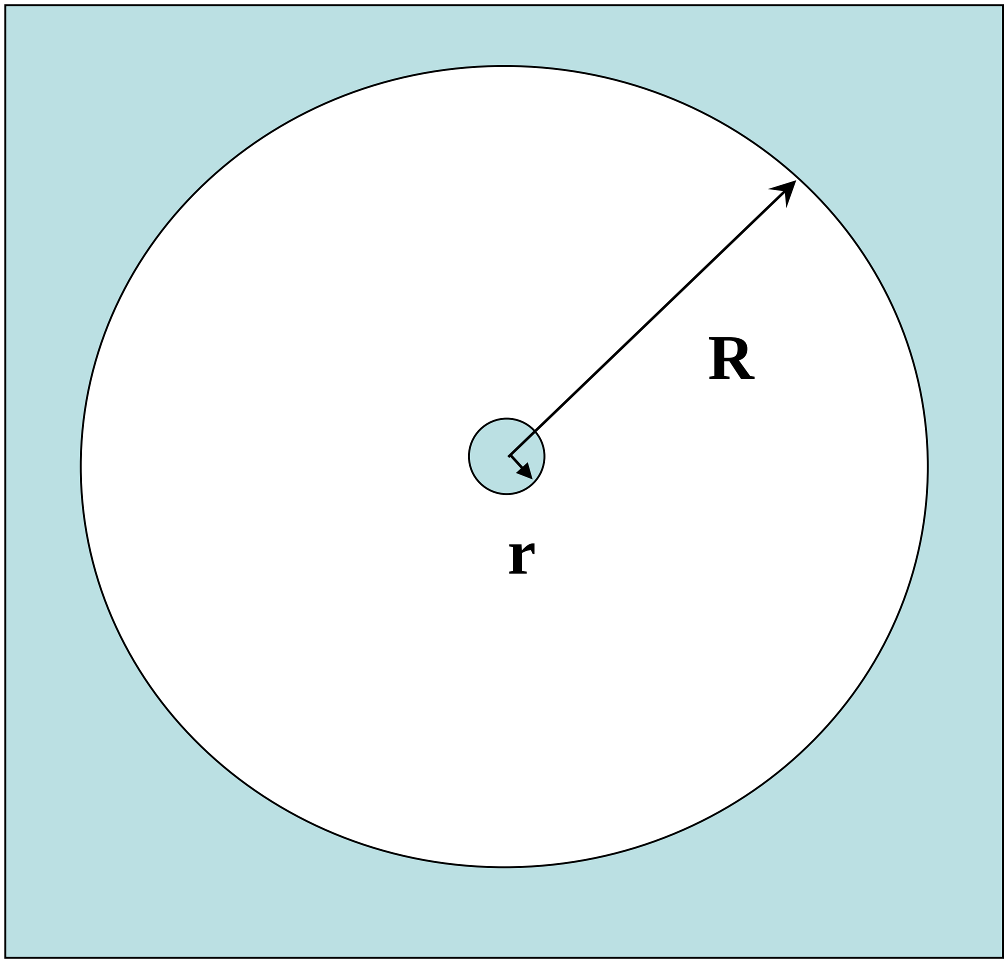
Disturbance of light F(ξ) on opaque disk is given by;

$$F(\xi) = \frac{1}{i \cdot \lambda \cdot f_{obj}} \int_r^R f(\eta) \exp\left\{ -\frac{i \cdot 2 \cdot \pi \cdot x \cdot \eta}{\lambda \cdot f_{obj}} \right\} d\eta$$

In here f(η) is disturbance of light on objective lens.

Objective lens diffraction

The integration performs r and R



r: radius of Anti-reflection disk

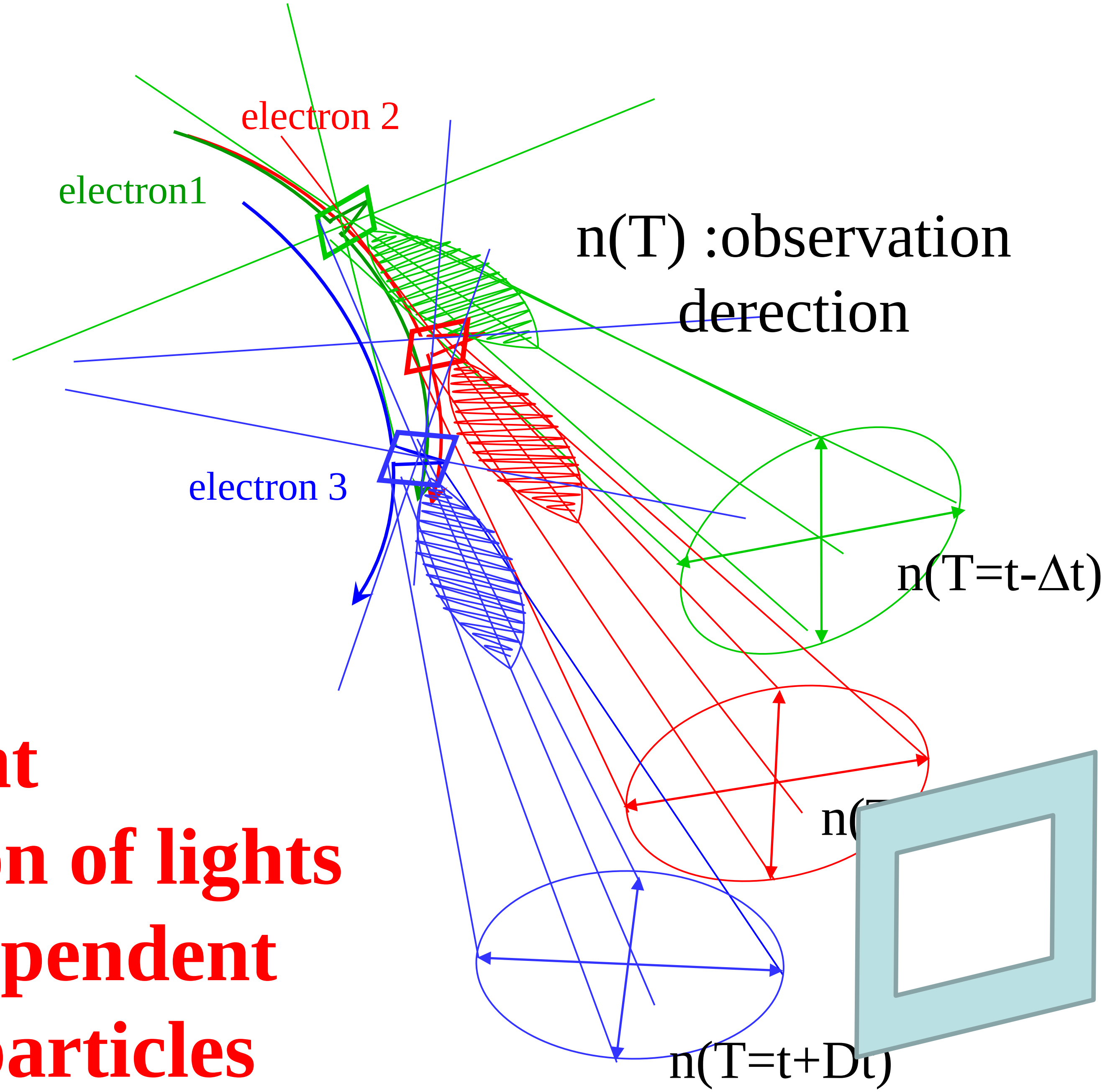
R: radius of objective lens aperture

Using Babinet's theorem (same technique in the appodization) , F(ξ) on opaque disk is given by

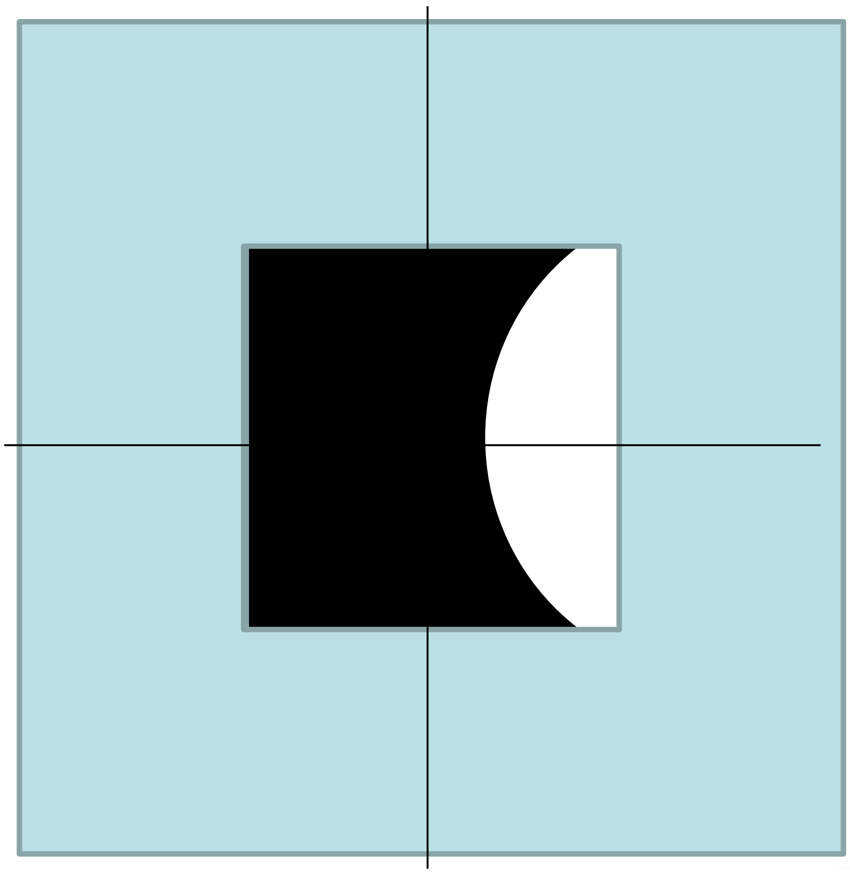
$$F(\xi)=\frac{1}{i\cdot\lambda\cdot f_{obj}}\left[\int_0^R f(\eta)\exp\left\{-\frac{i\cdot 2\cdot\pi\cdot x\cdot\eta}{\lambda\cdot f_{obj}}\right\}d\eta\right.$$

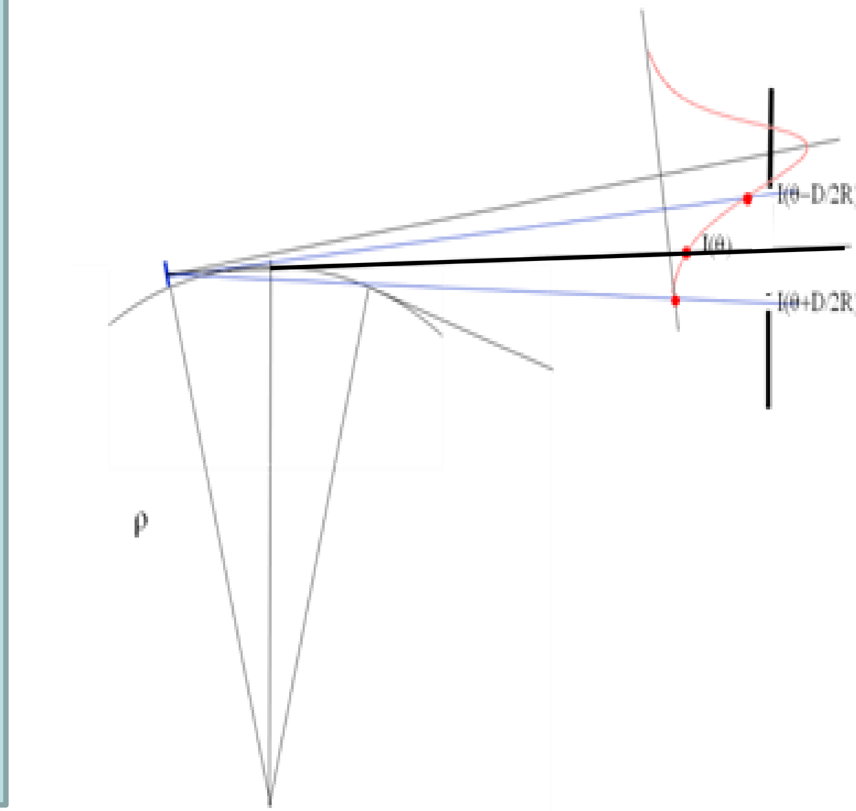
$$\left.-\int_0^r f(\eta)\exp\left\{-\frac{i\cdot 2\cdot\pi\cdot x\cdot\eta}{\lambda\cdot f_{obj}}\right\}d\eta\right]$$

Diffraction in horizontal direction



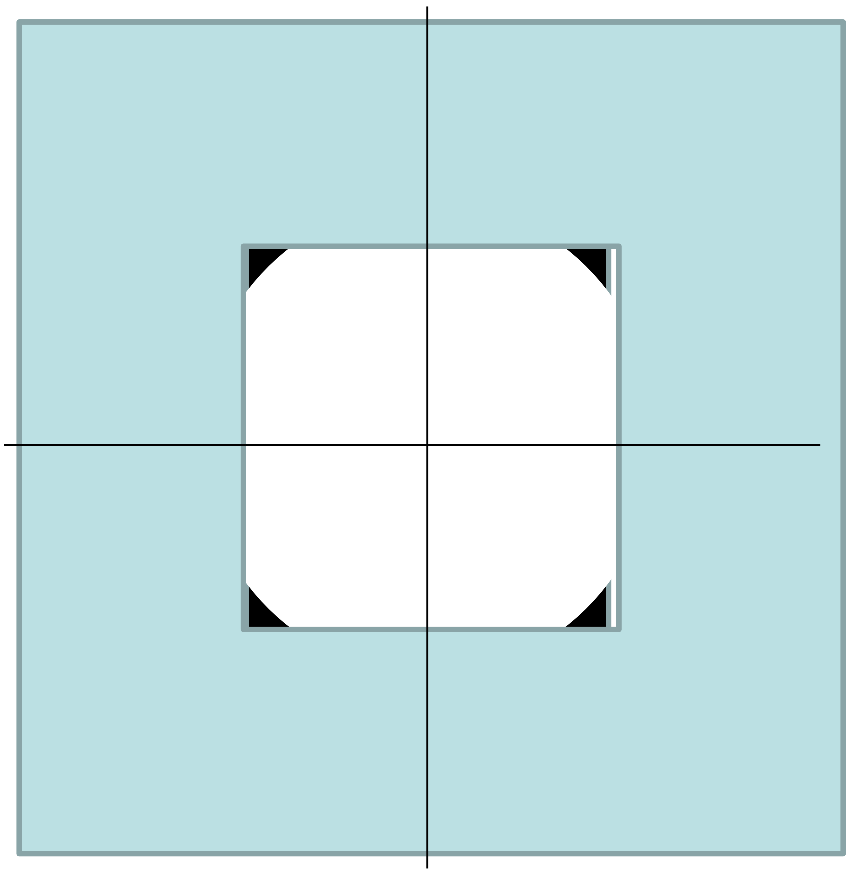
Incoherent summation of lights from independent charged particles

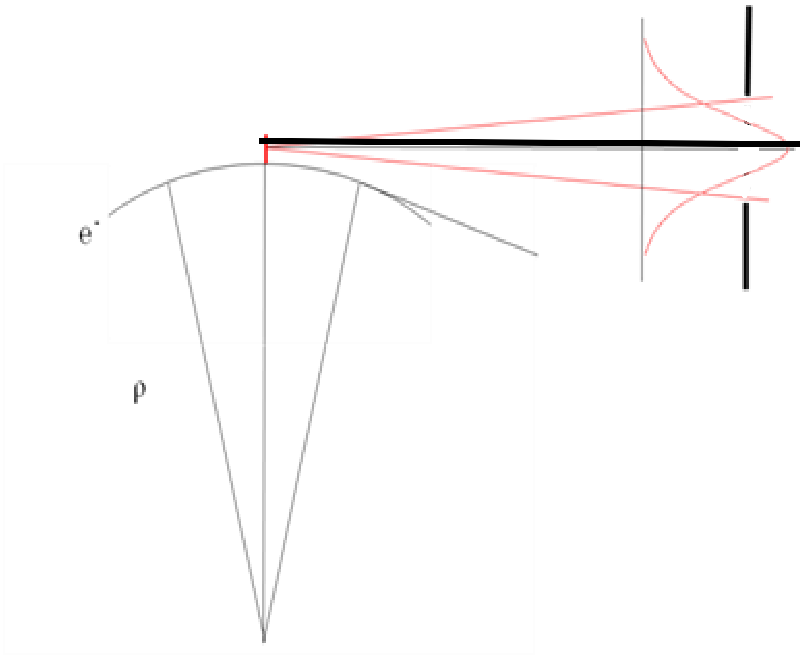


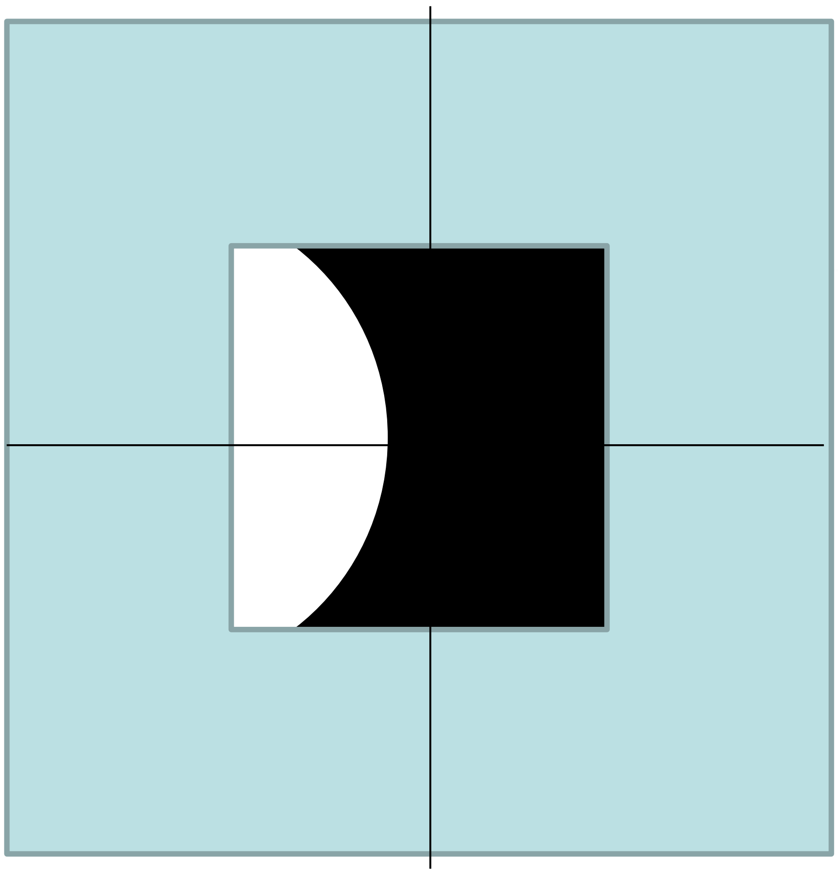


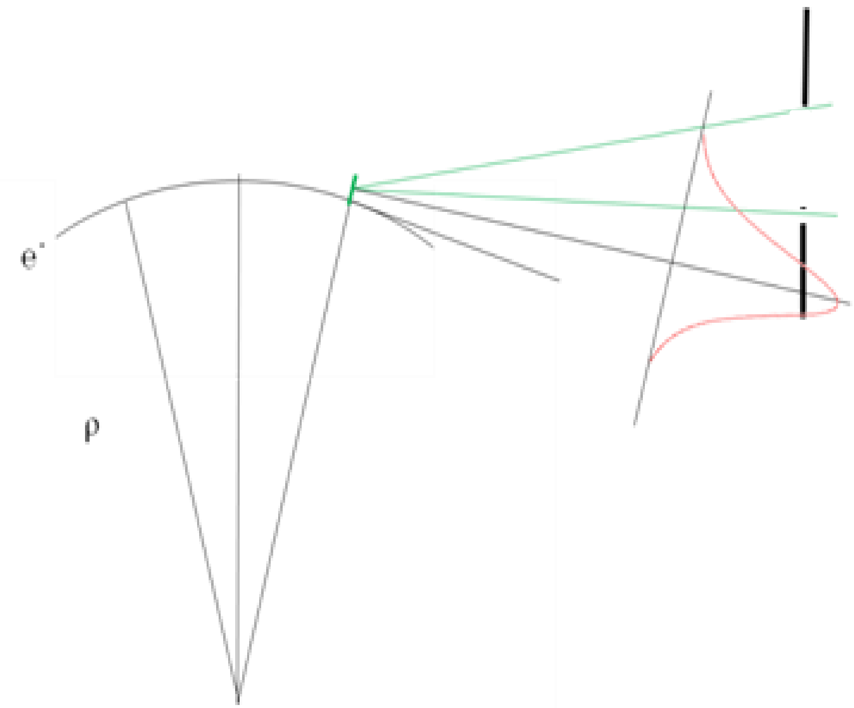
I(θ−D/2R)
I(θ)
I(θ+D/2R)
ρ

Full moon









e^-
ρ

Instantaneous diffraction pattern at focus point of Objective lens is given by,

$$\boldsymbol{F}\left(x_{obj},y_{obj},\theta\right)=\frac{1}{\boldsymbol{i}\cdot\lambda\cdot f_{\boldsymbol{obj}}}\iint F_0\boldsymbol{(}x+R\theta,y\boldsymbol{)}\,\mathbf{exp}\left\{-\frac{\boldsymbol{i}\cdot 2\cdot\pi\cdot\left(x\cdot x+y\cdot y\right)}{\lambda\cdot f_{\boldsymbol{obj}}}\right\}dx\,dy$$

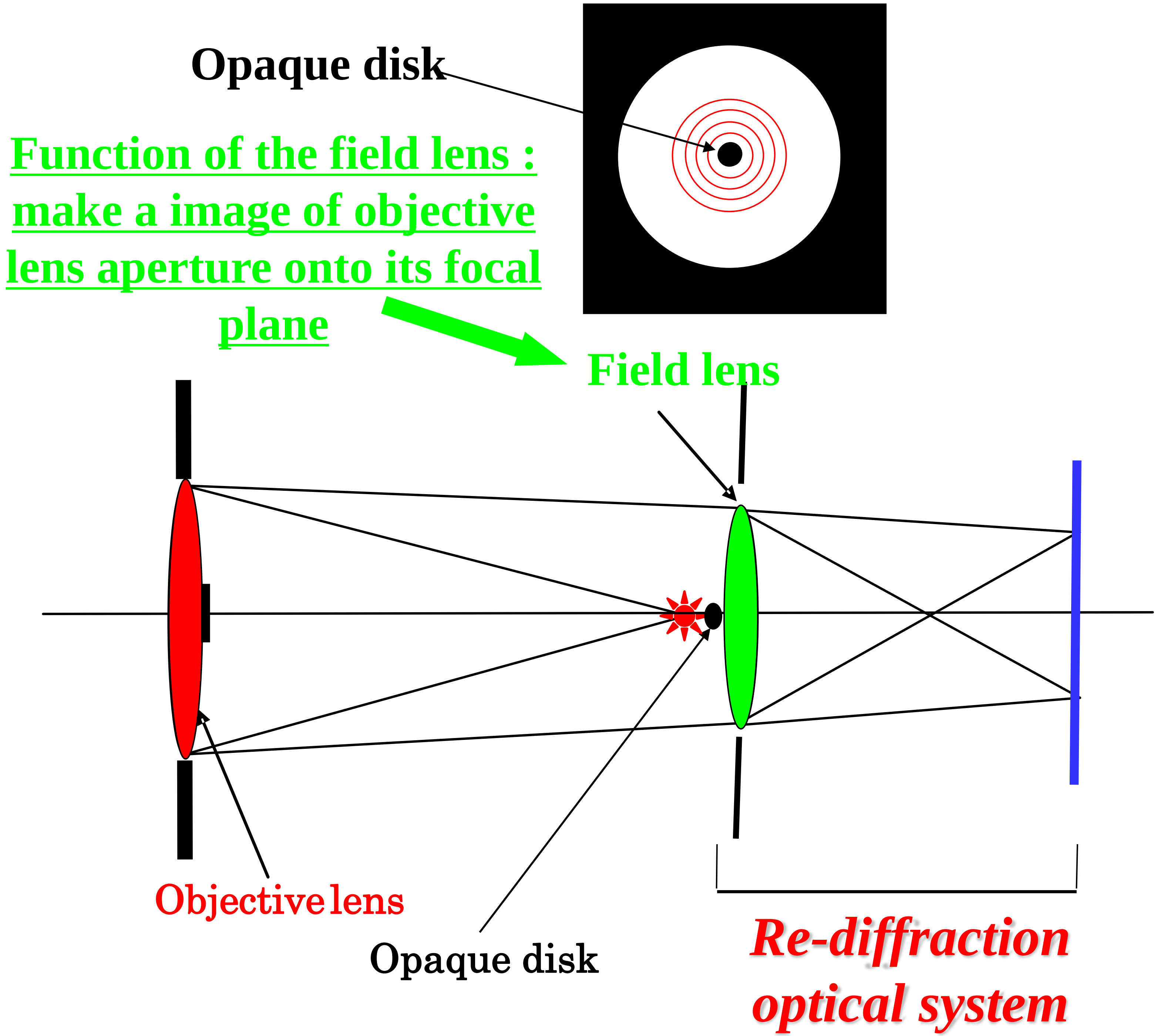
$$F_0(x,y)=\left\{\frac{e^2}{3\pi^2c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\left(\frac{x}{R}\right)^2+\left(\frac{y}{R}\right)^2\right)^2\left[K_{2/3}^2(\zeta)+\frac{\psi^2}{\frac{1}{\gamma^2}+\left(\frac{x}{R}\right)^2+\left(\frac{y}{R}\right)^2}K_{1/3}^2(\zeta)\right]\right\}^{\frac{1}{2}}$$

$$\zeta=\frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\left(\frac{x}{R}\right)^2+\left(\frac{y}{R}\right)^2\right)^{\frac{3}{2}}$$

Diffraction pattern on focus point is given by integrating instantaneous diffraction pattern in incoherent manner ,

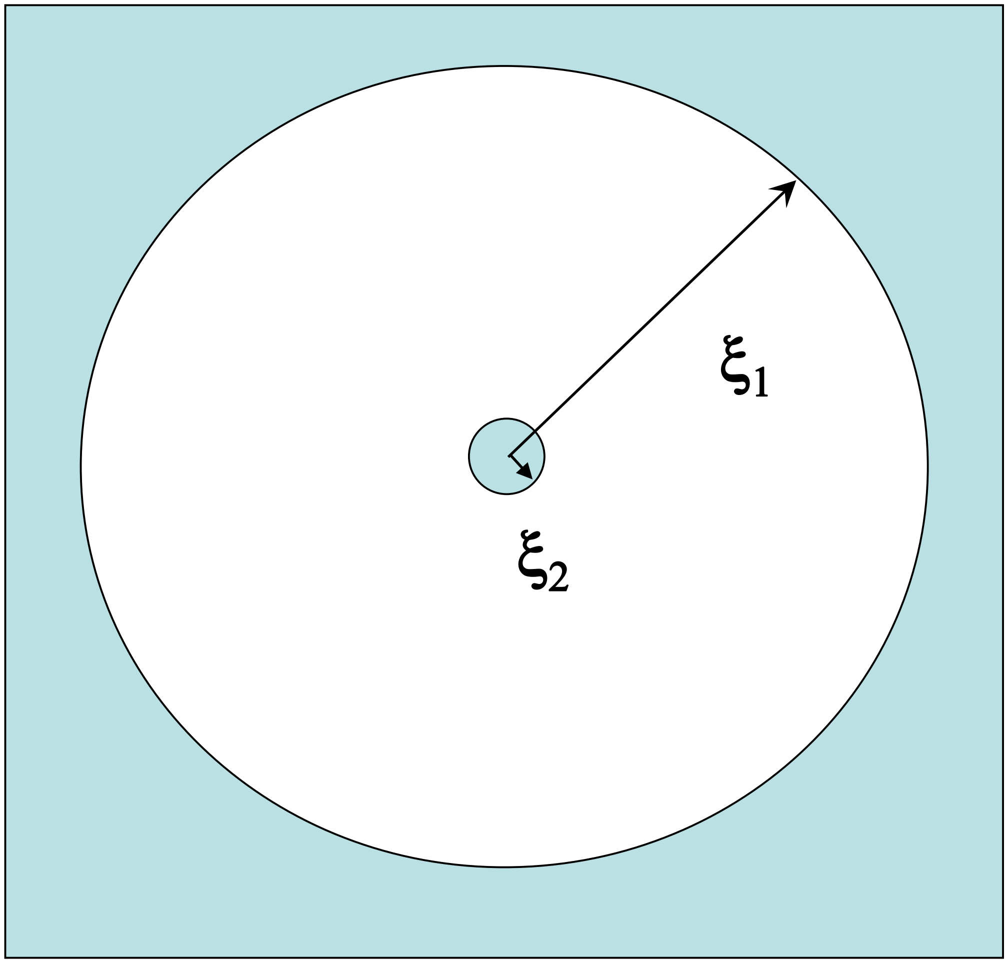
$$\boldsymbol{I}_{\boldsymbol{obj}}\left(x_{obj},y_{obj}\right)=\int\left|\boldsymbol{F}^2\left(x_{obj},y_{obj},\theta\right)\right|d\theta$$

2. Re-diffraction system


Opaque disk
Function of the field lens : make a image of objective lens aperture onto its focal plane
Field lens
Objective lens
Opaque disk
Re-diffraction optical system

Field lens diffraction

The integration performs ξ_1 and ξ_1



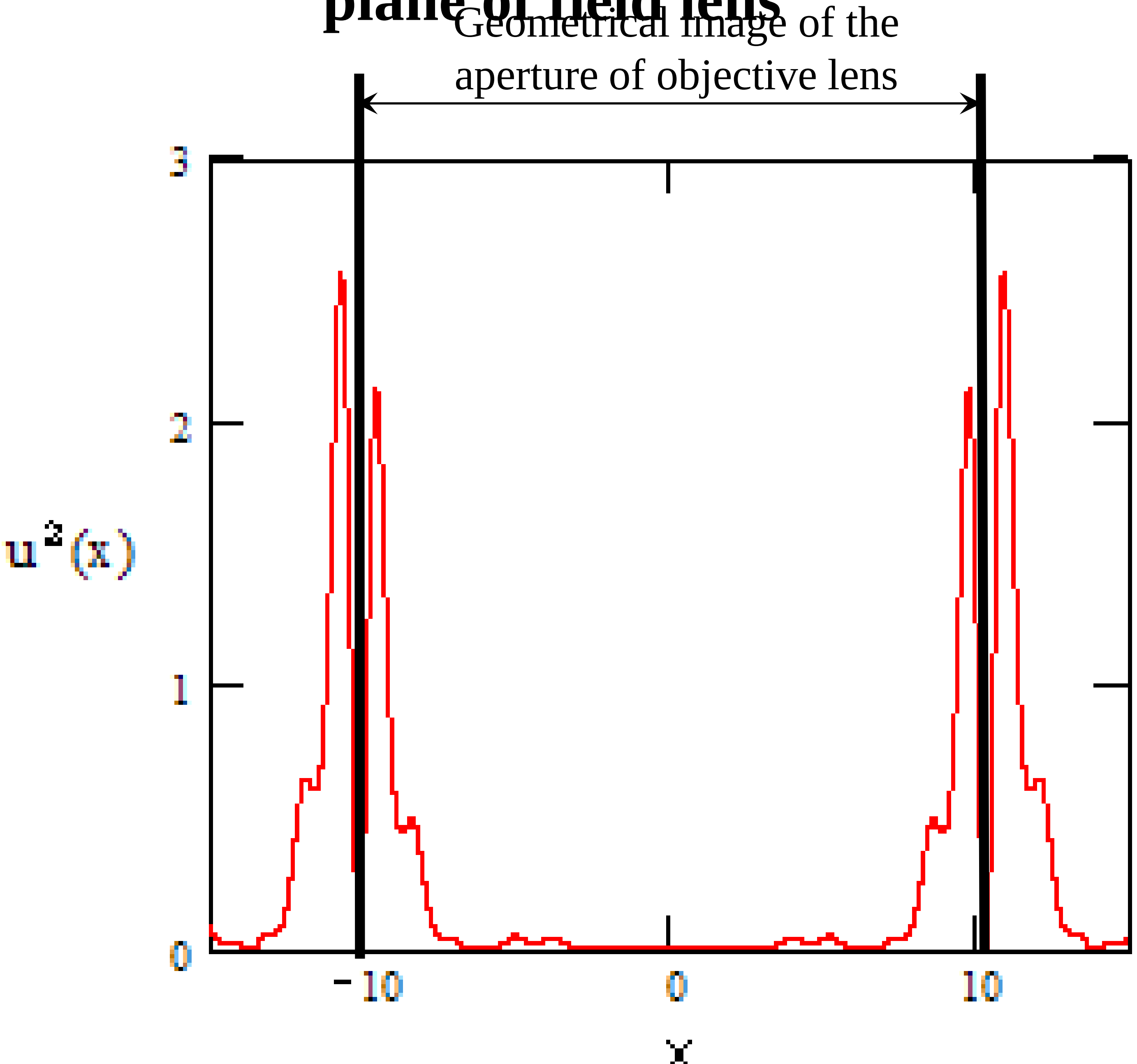
ξ_1: radius of field lens

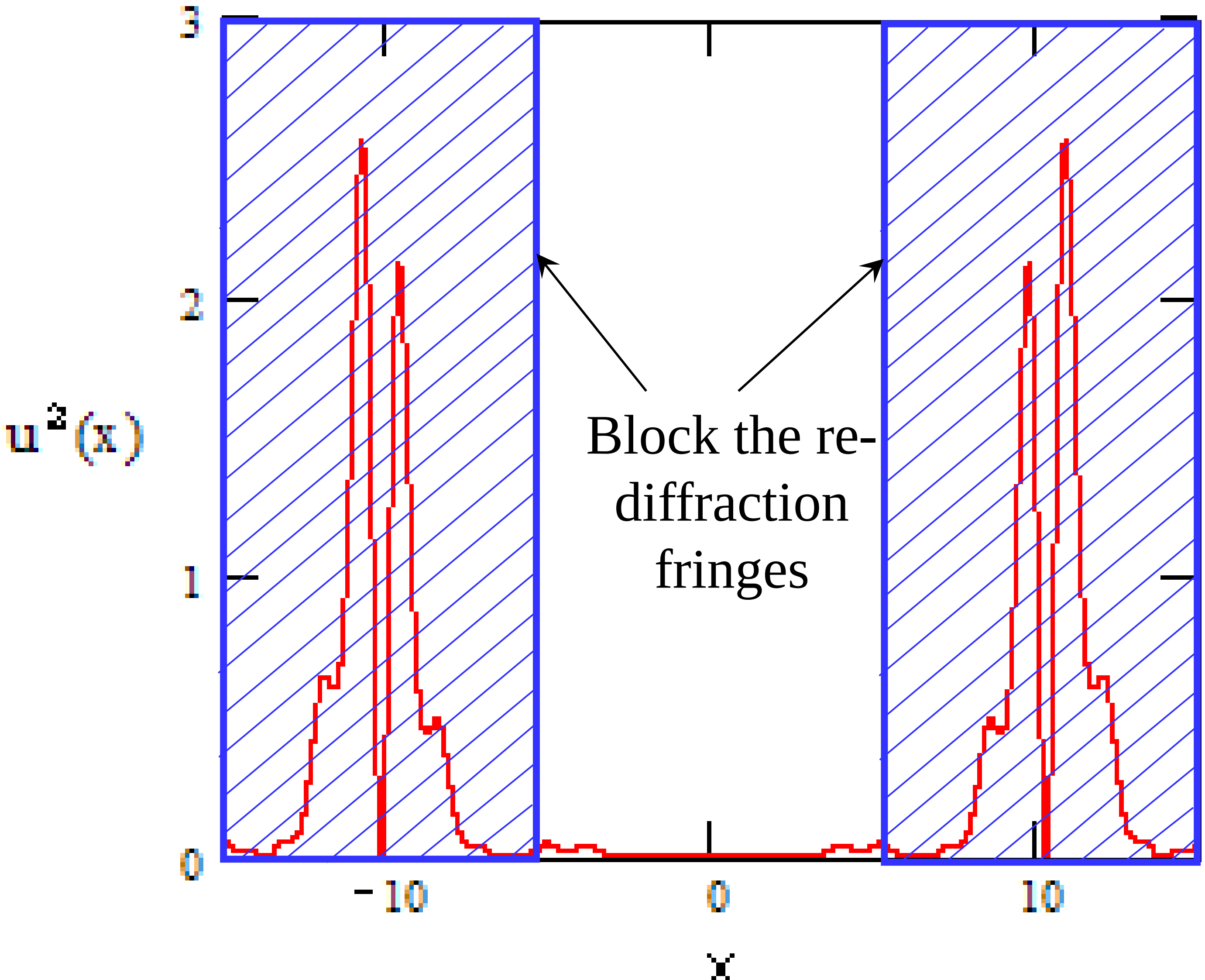
ξ_2: radius of opaque disk

Disturbance of light on Lyot's stop by re-diffraction system is given by;

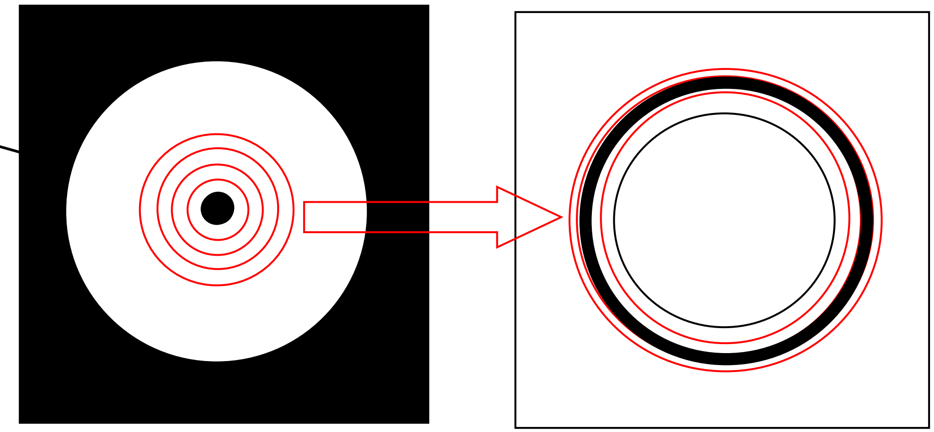
$$u(x)=\frac{1}{i\cdot\lambda\cdot f_{\mathrm{field}}}\int_{\xi 1}^{\xi 2}F(\xi)\exp\left\{-\frac{i\cdot 2\cdot\pi\cdot x\cdot\xi}{\lambda\cdot f_{\mathrm{field}}}\right\}d\xi$$

Intensity distribution of diffraction fringes on focus plane of field lens

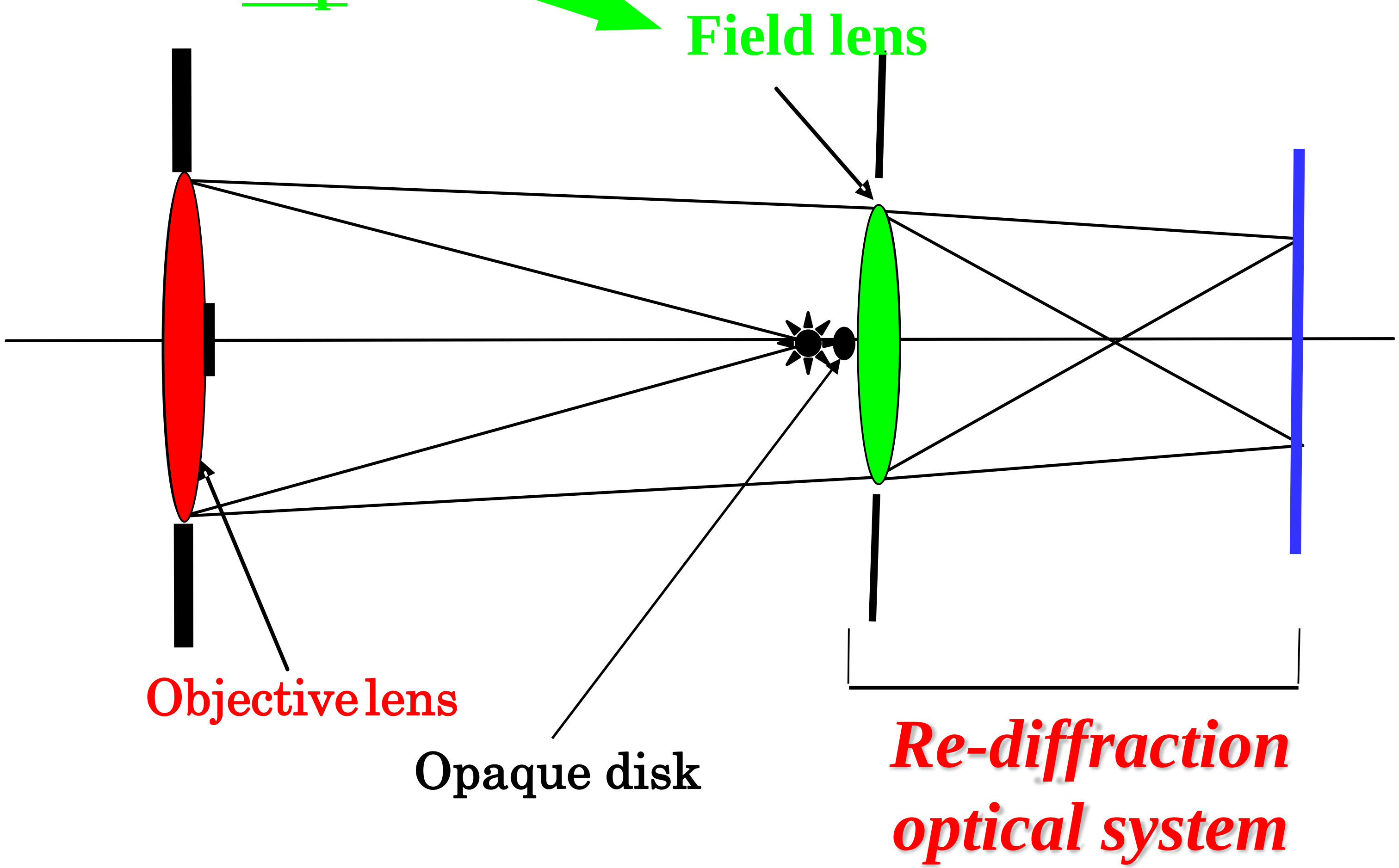


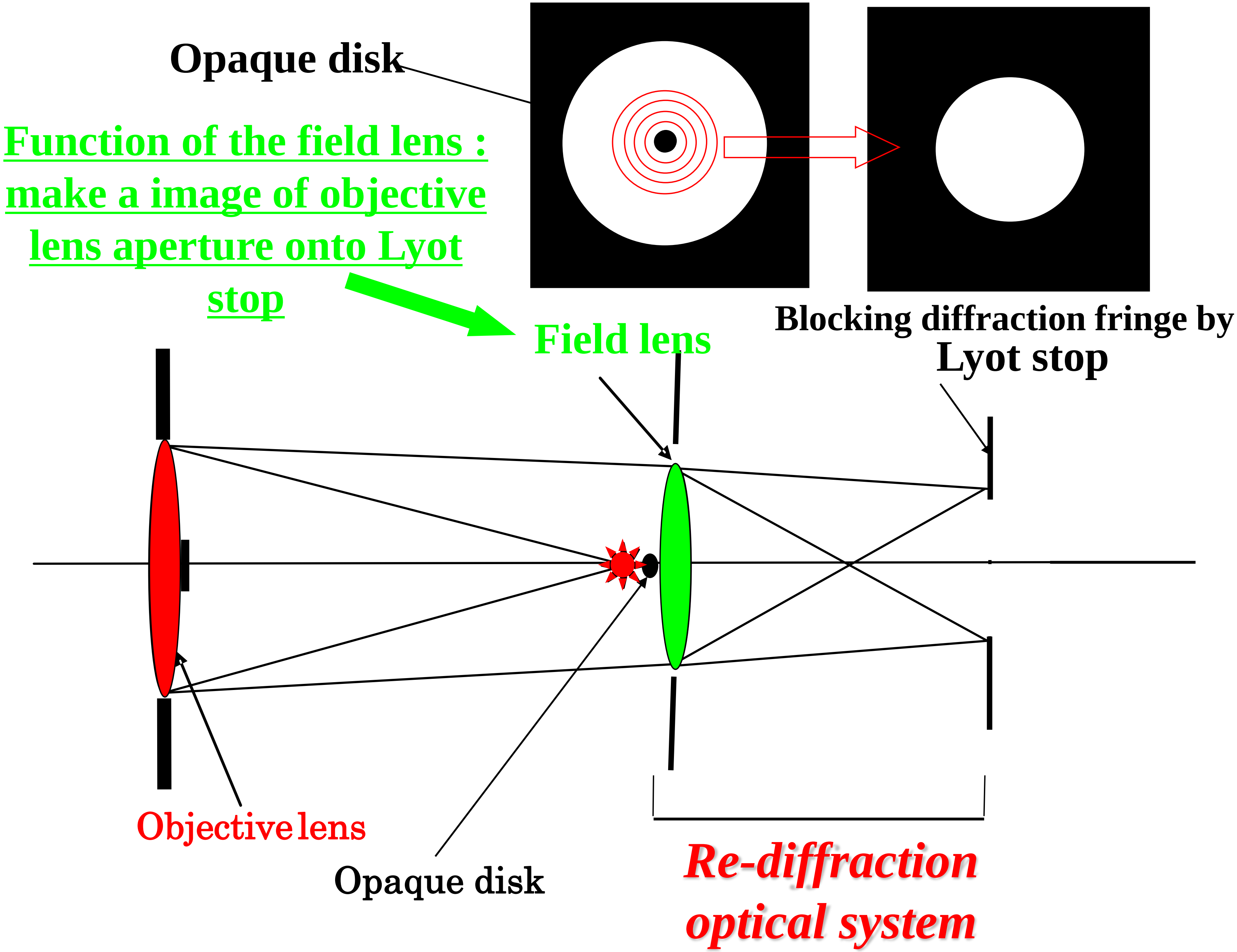


3
2
1
0
$u^2(x)$
-10
0
10
x
Block the re-
diffraction
fringes



Function of the field lens : make a image of objective lens aperture onto Lyot stop



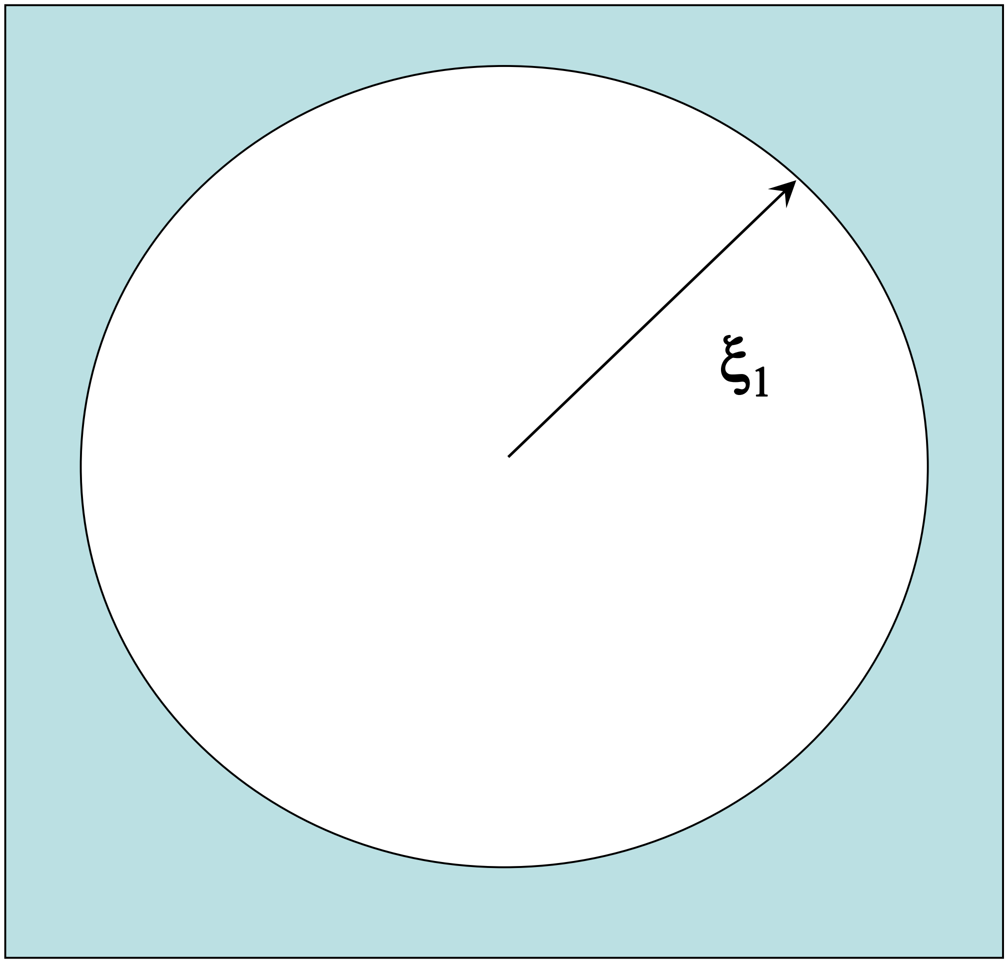


Opaque disk
Function of the field lens : make a image of objective lens aperture onto Lyot stop
Field lens
Blocking diffraction fringe by Lyot stop
Objective lens
Opaque disk
Re-diffraction optical system

3. Relay lens

Relay lens diffraction

The integration performs η_1



η_1: radius of

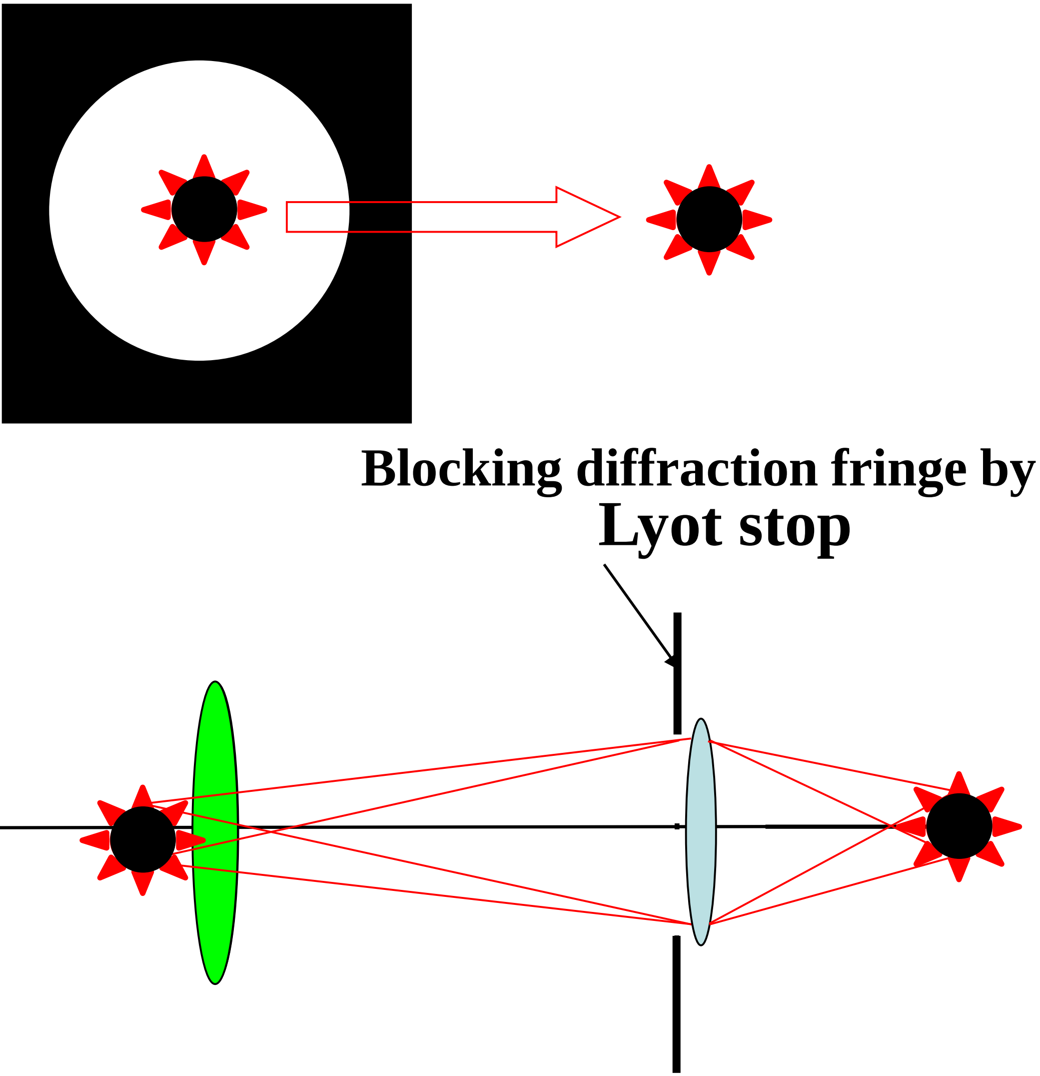
Lyot stop

Disturbance of light on final focus point V(x) is given by;

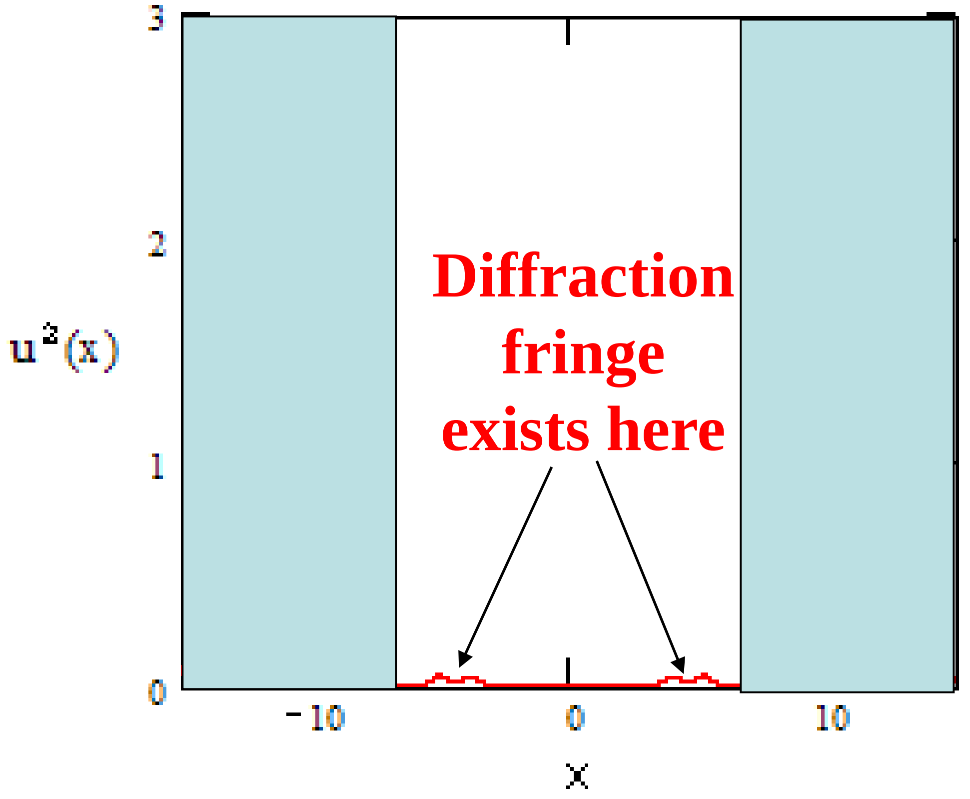
$$V(\phi) = \frac{1}{i \cdot \lambda \cdot f_{relay}} \int_0^{\phi 1} u(x) \exp\left\{ -\frac{i \cdot 2 \cdot \pi \cdot \phi \cdot x}{\lambda \cdot f_{relay}} \right\} dx$$

U(x) is still not 0 inside of relay lens pupil!

Relay of corona image to final focus point



Background in classical coronagraph



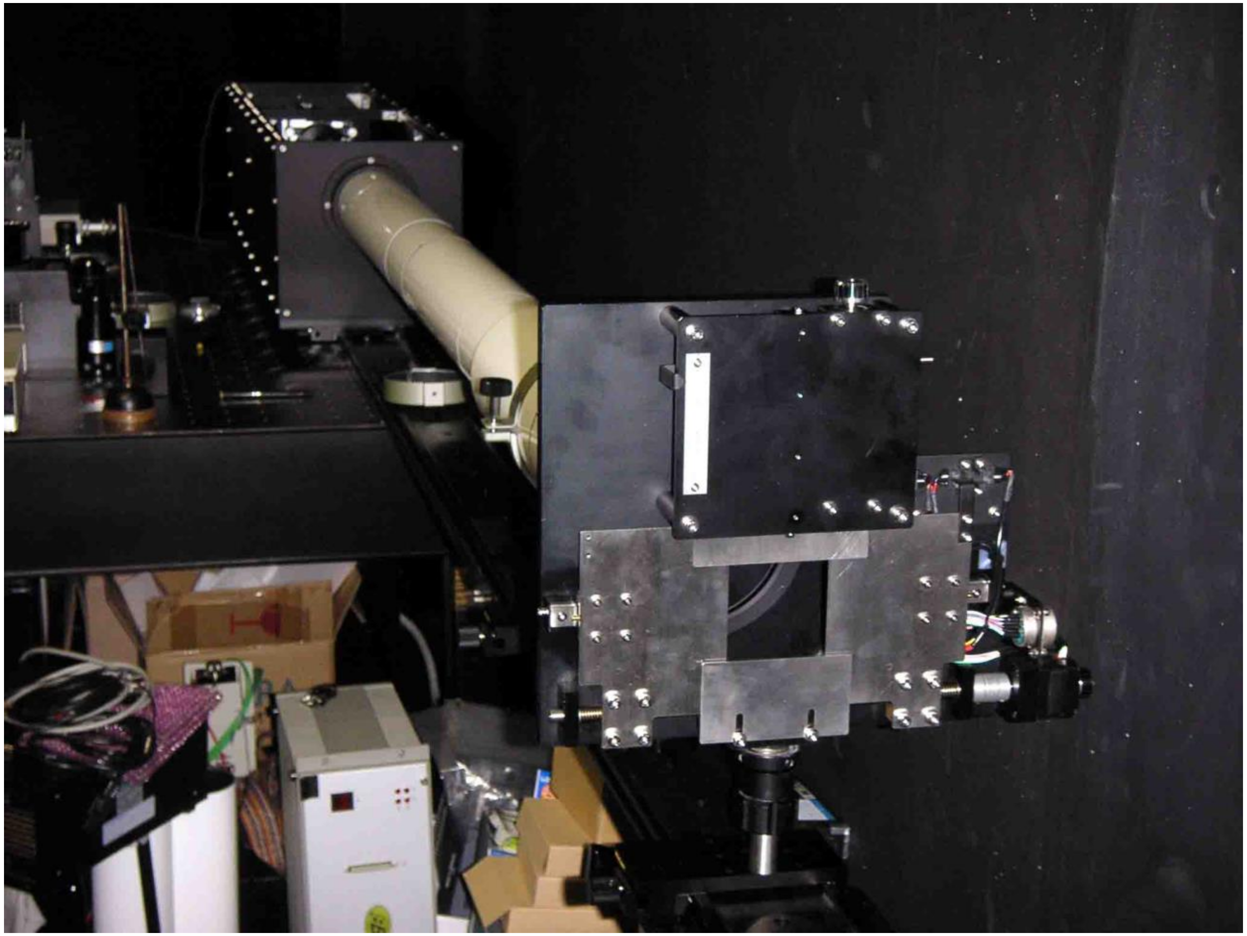
This leakage of the diffraction fringe can make background level $10^{-8 \text{ to } -9}$ (depends on Lyot stop condition).

Re-diffraction intensity on the Lyot stop

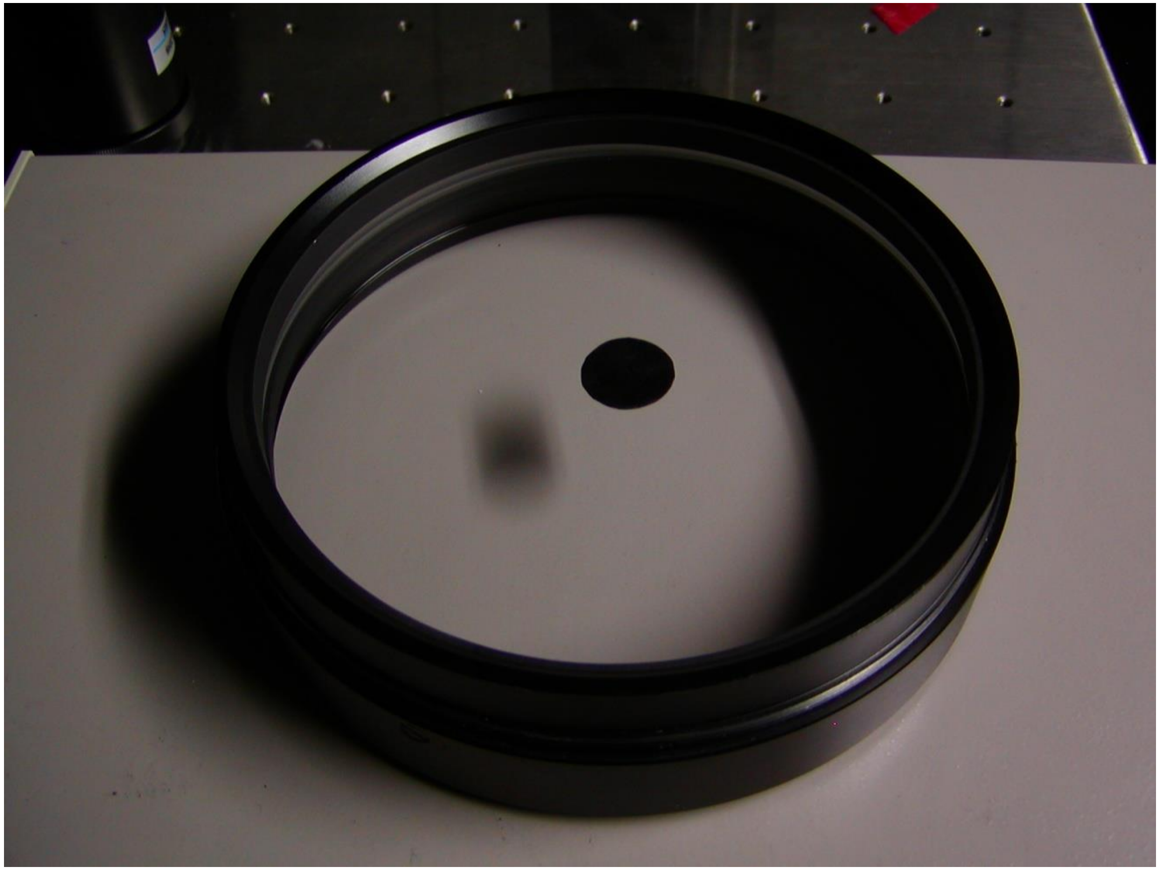
Observation of the sun corona by coronagraph

Photographs of coronagraph

Front view of the coronagraph



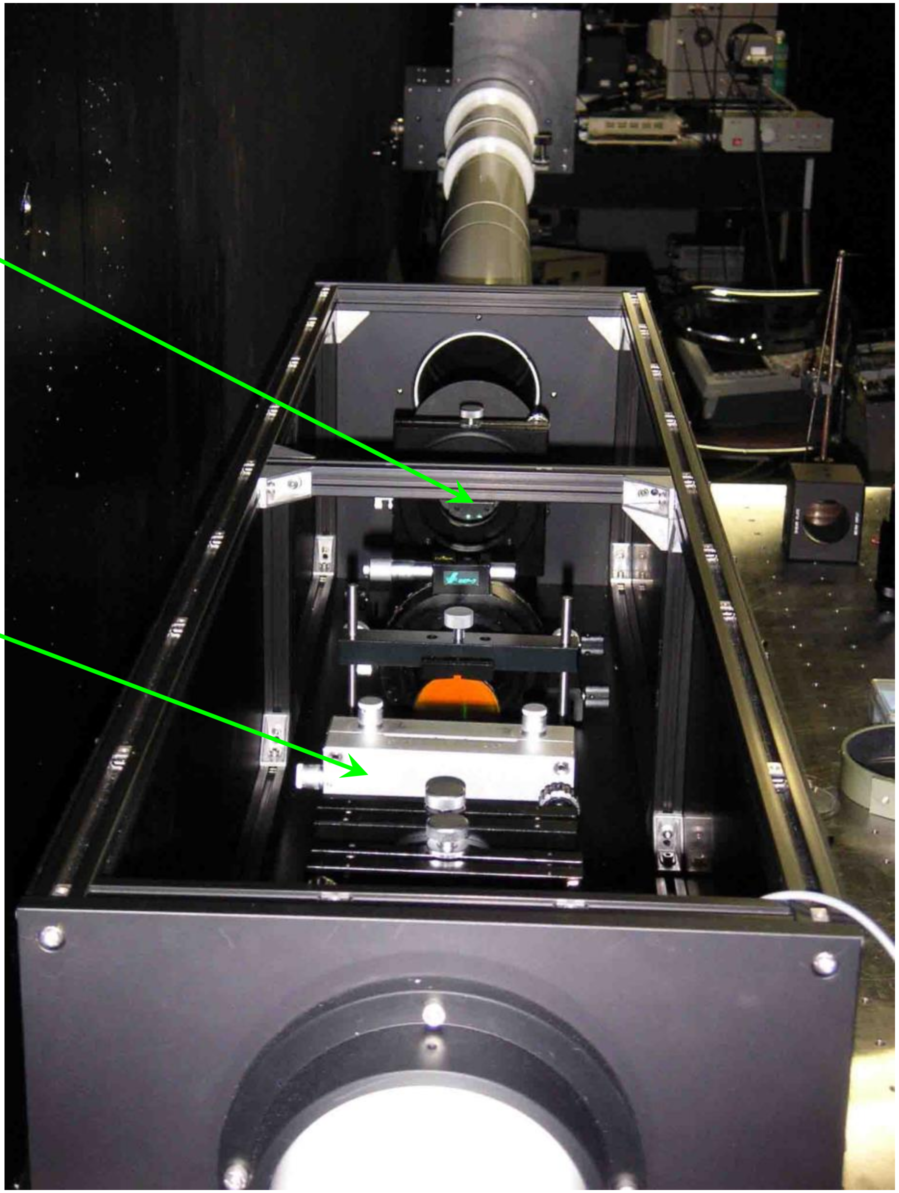
Objective lens with anti-reflection disk to block reflected light from opaque disk



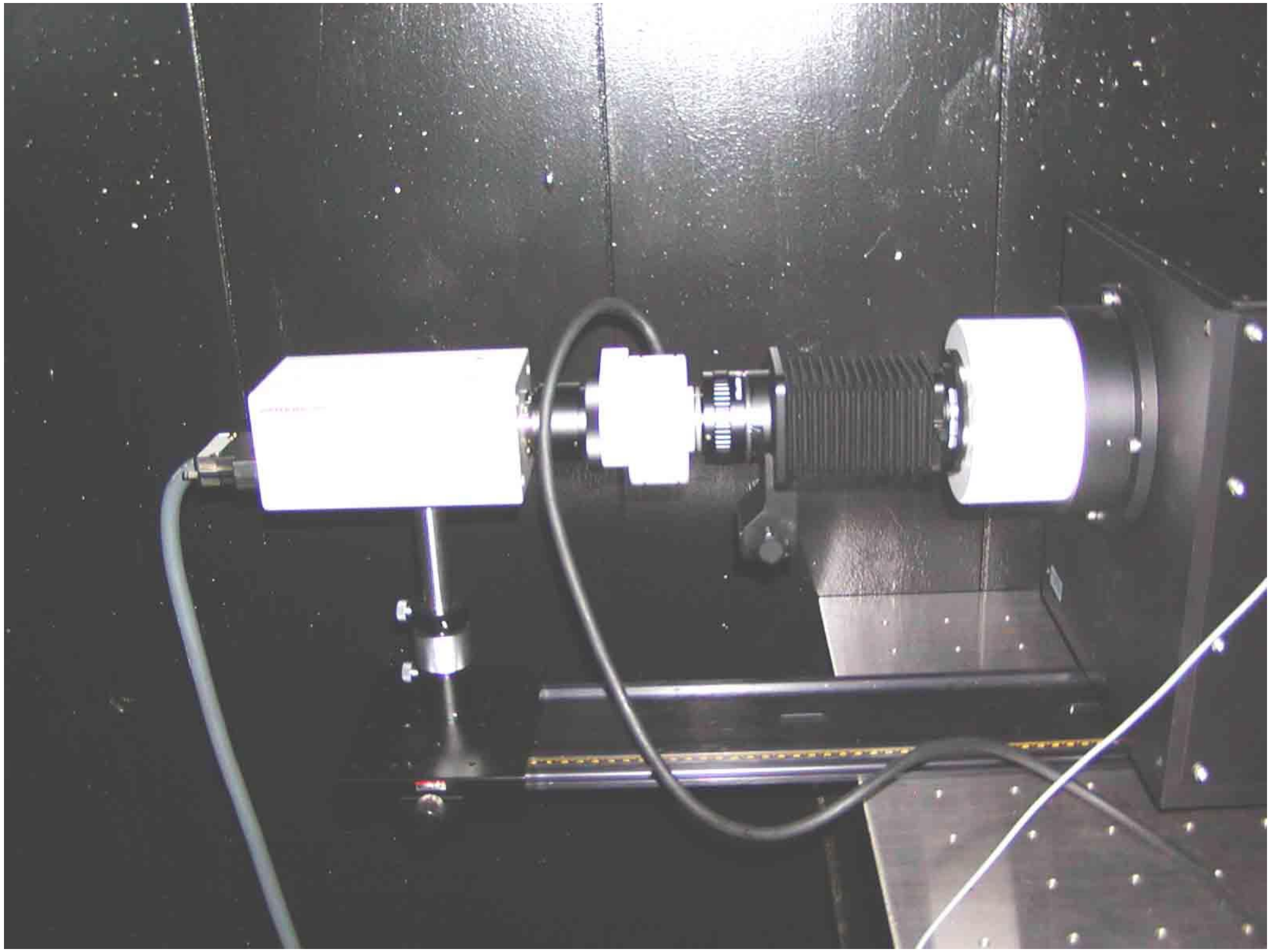
View from the back side

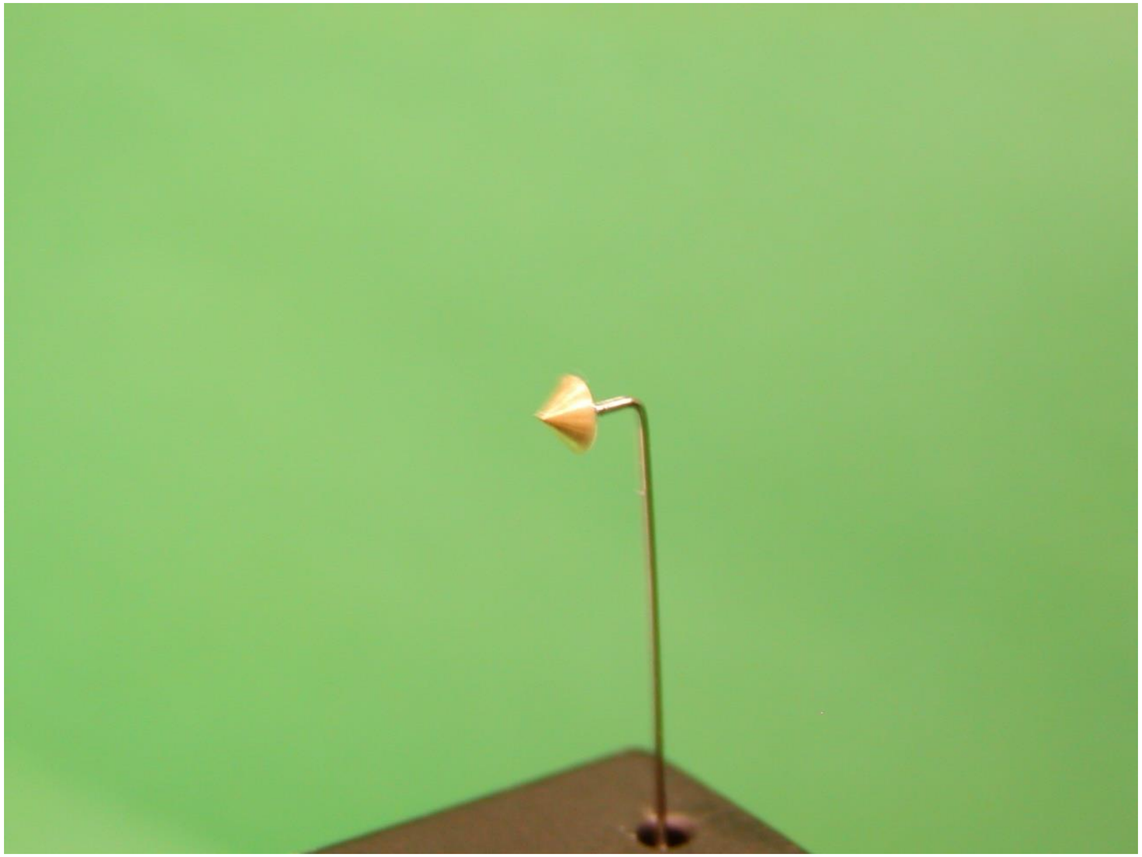
Field lens

Lyot's stop



Fast gated camera set on the final focusing point

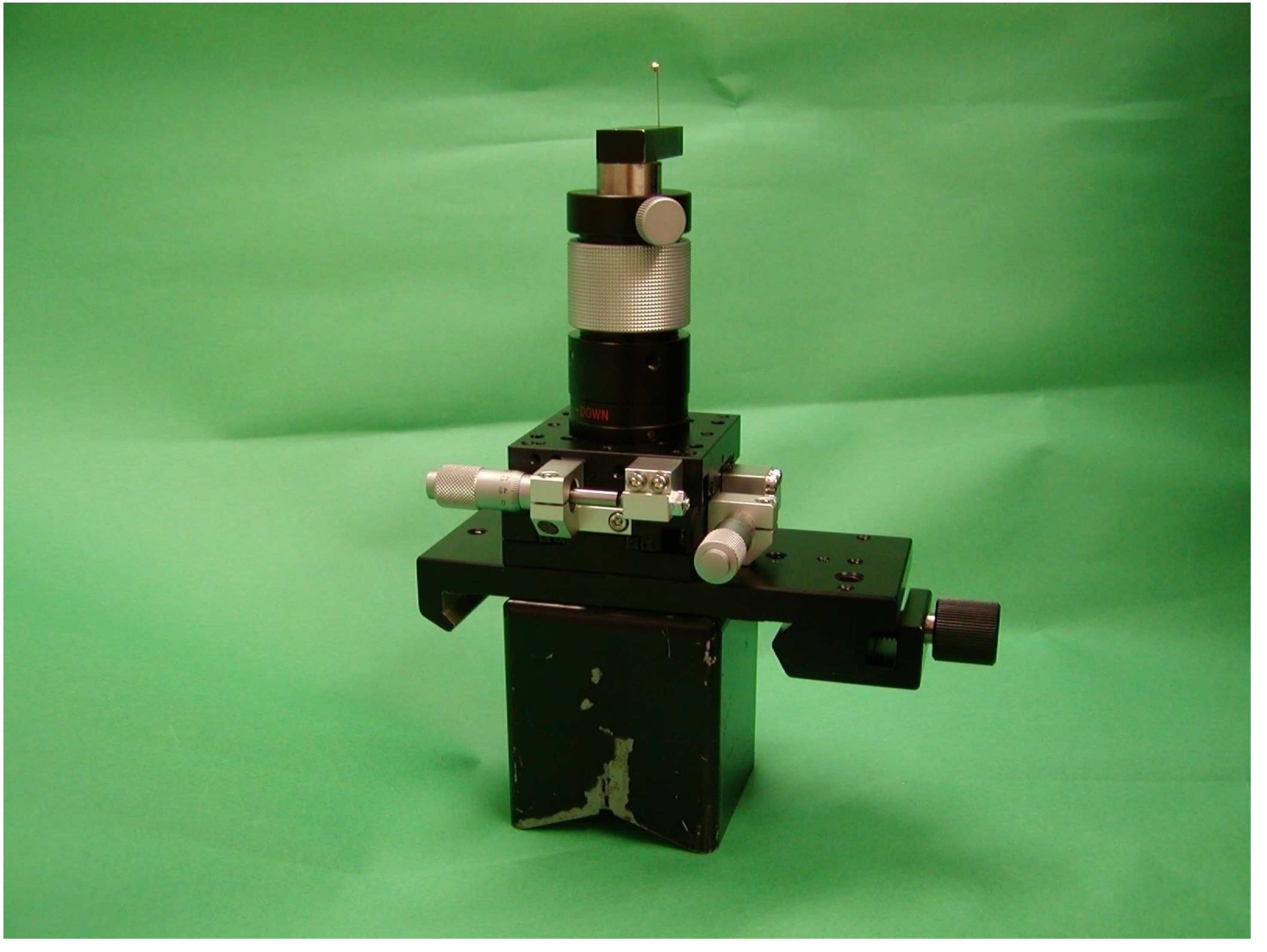




Zoom up of opaque disk.

Shape is cone and top-angle is 90º

Opaque disk assembly



Background source in coronagraph

1. **Noise from defects on the lens surface (inside) such as scratches and digs.**

2. **Noise from the optical components (mirrors) in front of the coronagraph.**

3. **Reflections in inside wall of the coronagraph.**

4. **Noise from dust in air.**

Background source in coronagraph

1. **Noise from defects on the lens surface (inside) such as scratches and digs.**

2. **Noise from the optical components (mirrors) in front of the coronagraph.**

3. **Reflections in inside wall of the coronagraph. Apply baffles**

4. **Noise from dust in air. Apply dust filter**

S&D 60/40 surface of the lens

5mm

Noise source in the pupil has two effects,

1. **Refraction in noise source**

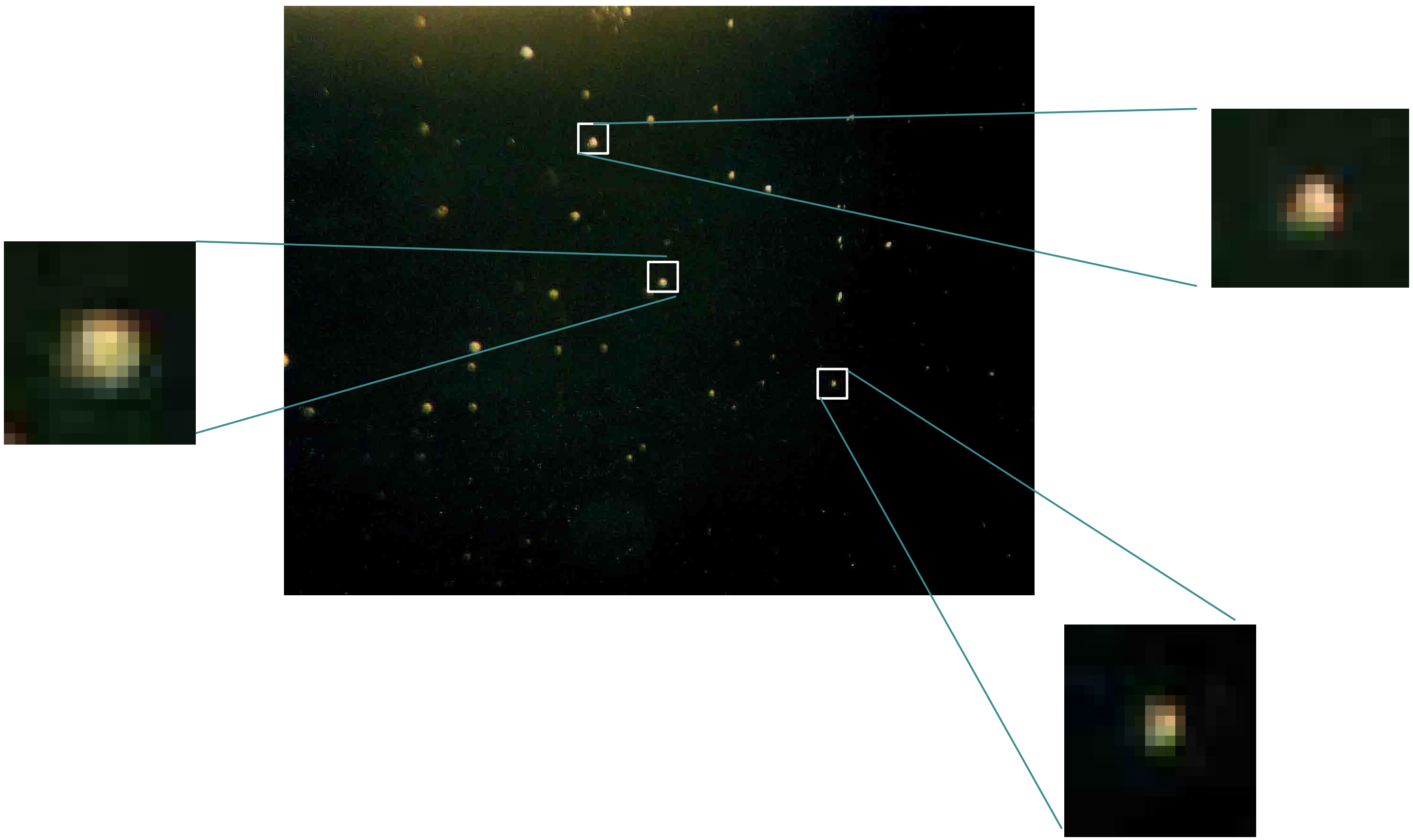
phenomena in sun or moon halo is produced by this effect.

2. **Mie scattering by noise source**

opaque dust or dig on lens or on any optical components are classified this effect.

Dust in air also has same effect.

Effect 2: Digs on glass surface serves as diffraction source in the pupil



A Big problem is

the nose sources produce ***<u>another halo</u>*!!**

A Big problem is

the nose sources produce
***another halo*!!**

It is not easy to distinguish from intrinsic beam halo

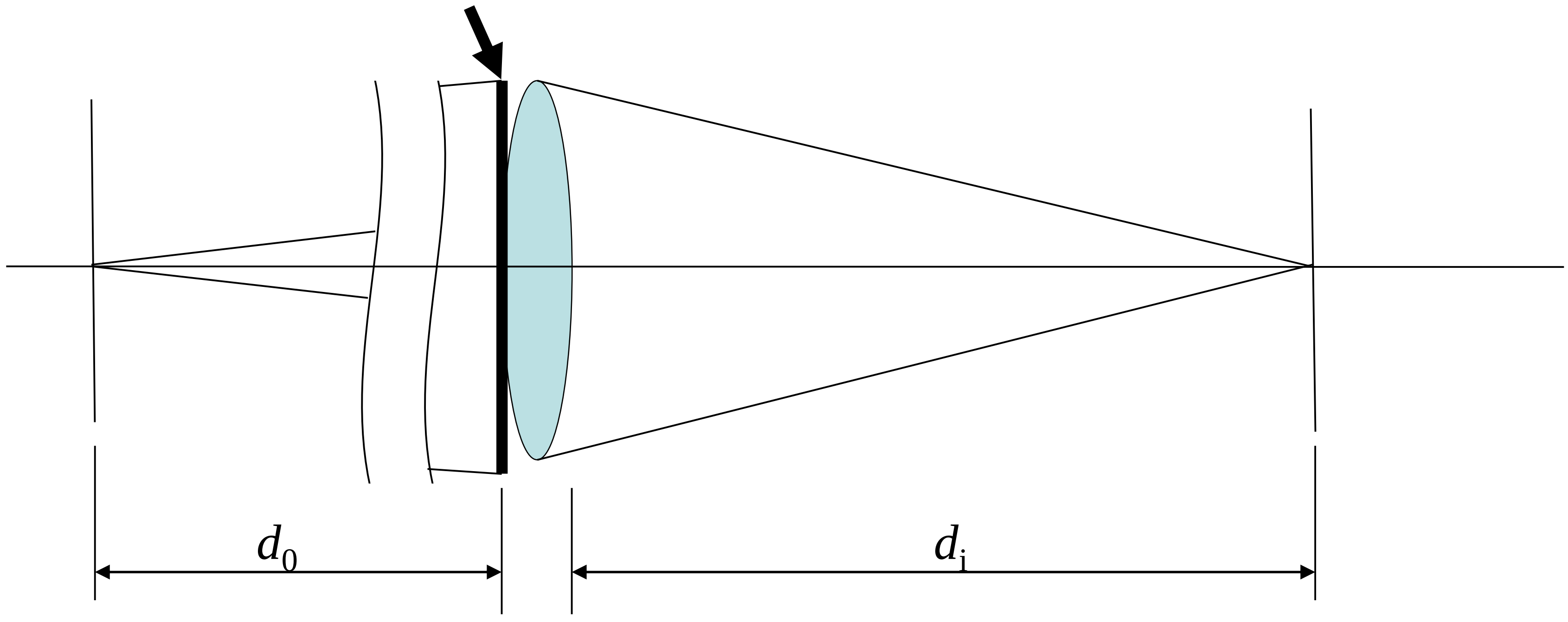
Mie scattering,

it's diffraction treatment

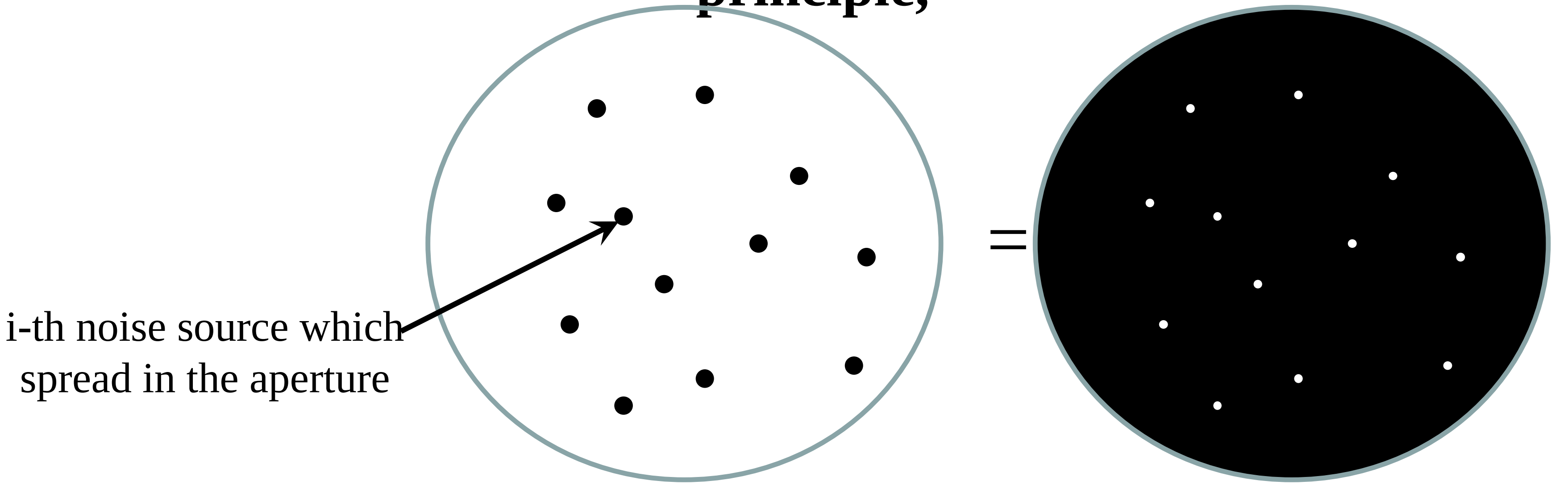
(following treatment, see Goodman p.83-96)

Case 1. Noise source in the entrance pupil of objective lens

***P*(x, y): pupil function with assembly of diffraction noise sources on the lens**



Let us approximate i-th noise source in the pupil as a opaque disk having a diameter of r_0 , Using the Babinet's principle,



$$P_i\left(r_0, x, y\right) = circ\left(r_0, x_i, y_i\right)$$

Then pupil function having many noise source is given by,

$$P\left(\bar{r}, x, y\right) = \sum_i P_i\left(r_{0,} x, y\right) \cdot \exp\left(-ik\left(x_i + y_i\right)\right)$$

When the mean distance of noise source is longer than 1st order transverse coherent length , pupil function with noise sources is simply given by,

$$P(\bar{r}, x, y) = \sum_i P_i(r_0, x, y)$$

Then the impulsive response $h(x_i, y_i; x_0, y_0)$ **on the image plane is given by,**

$$h(x_i, y_i; x_0, y_0) = \frac{1}{\lambda d_0 d_i} \iint P(\bar{r}, x, y) \exp\left\{-i \frac{2\pi}{\lambda d_i} \left[(x_i + Mx_0)x + (y_i + My_0)y\right]\right\} dxdy$$

in here, $M = d_i / d_0$ **denotes geometrical magnification.**

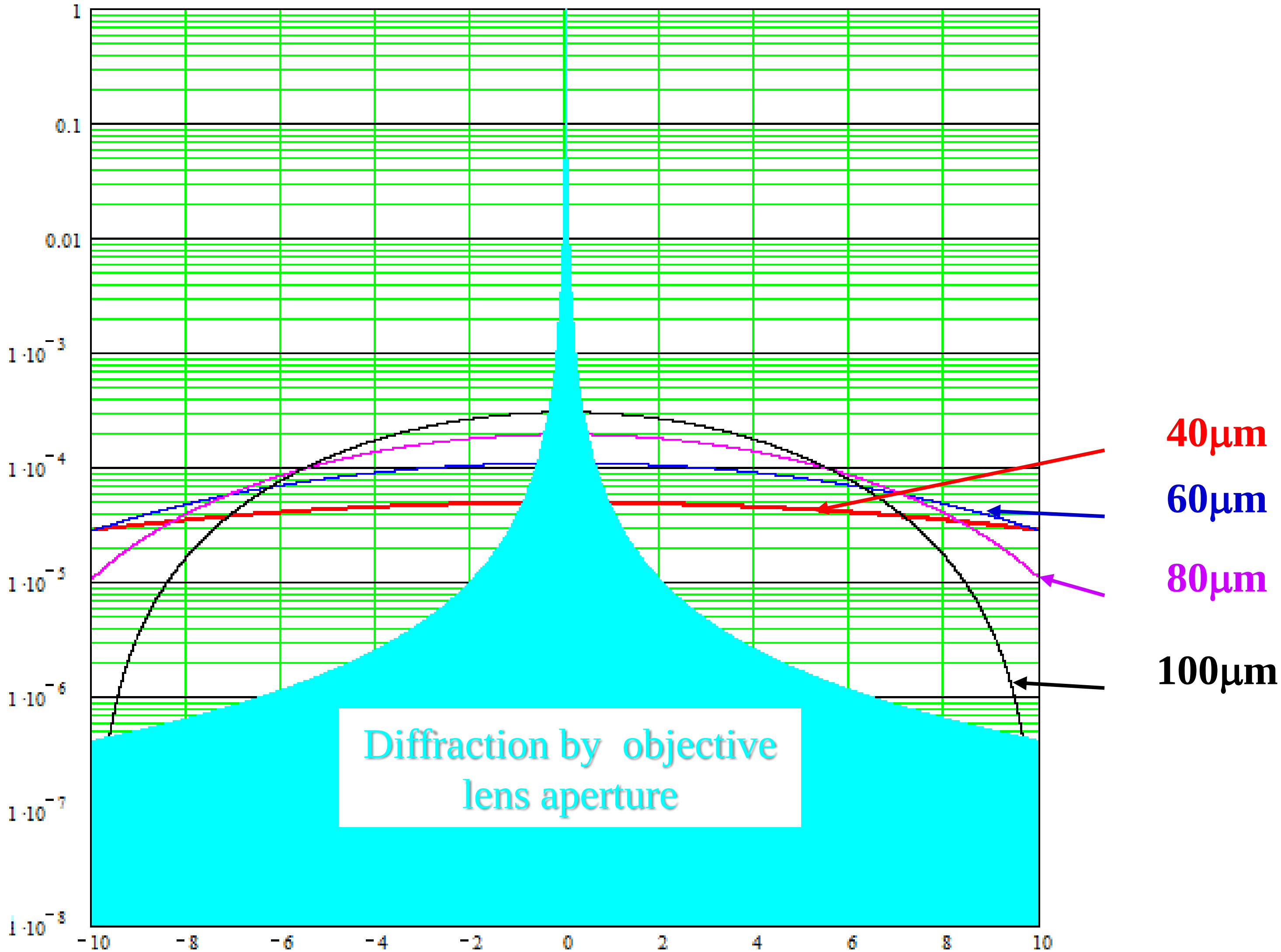
The intensity of diffraction from noise sources is inverse-proportional to extinction rate,

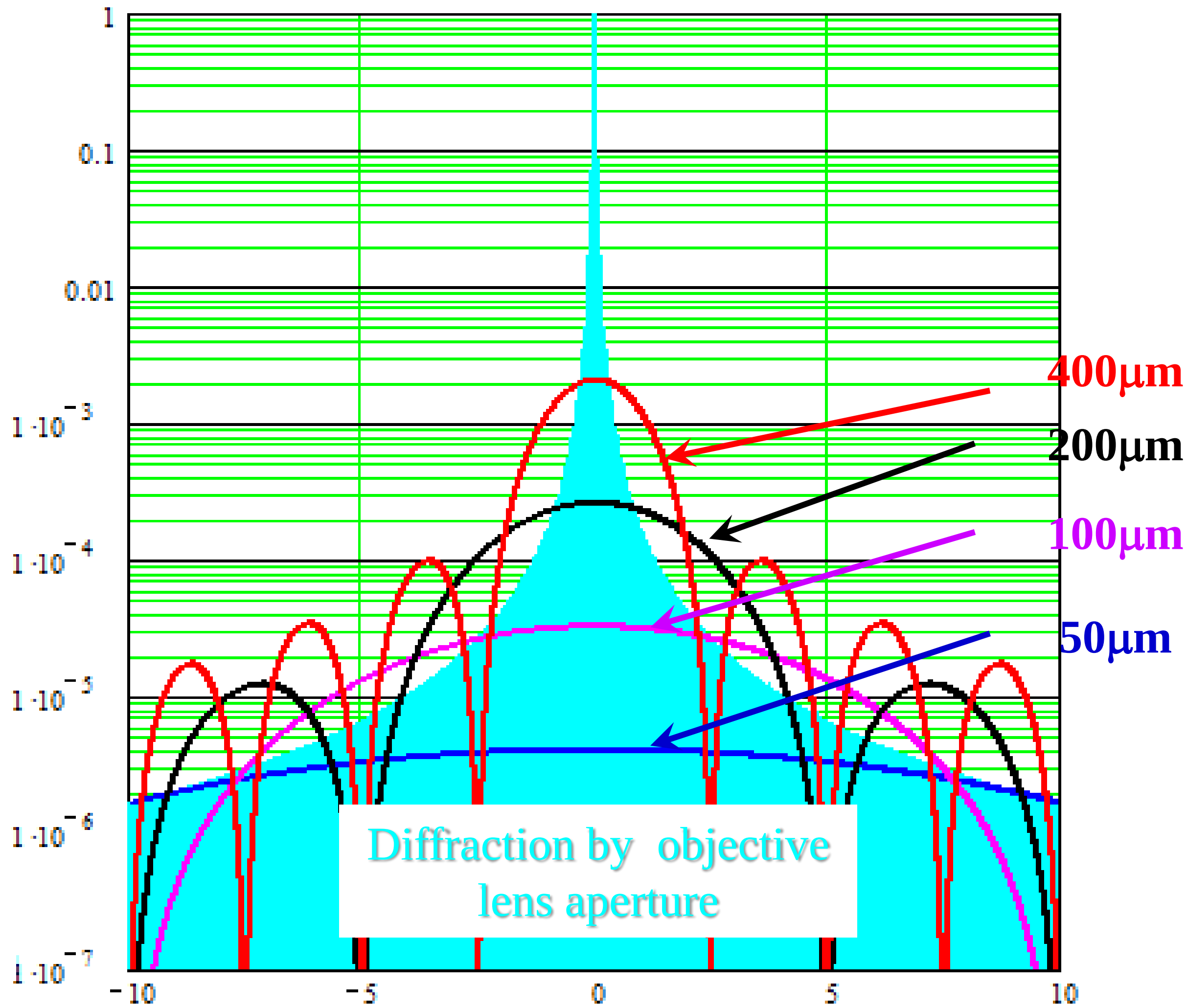
Extinction rate = entrance pupil aperture area

/ total area of noise source

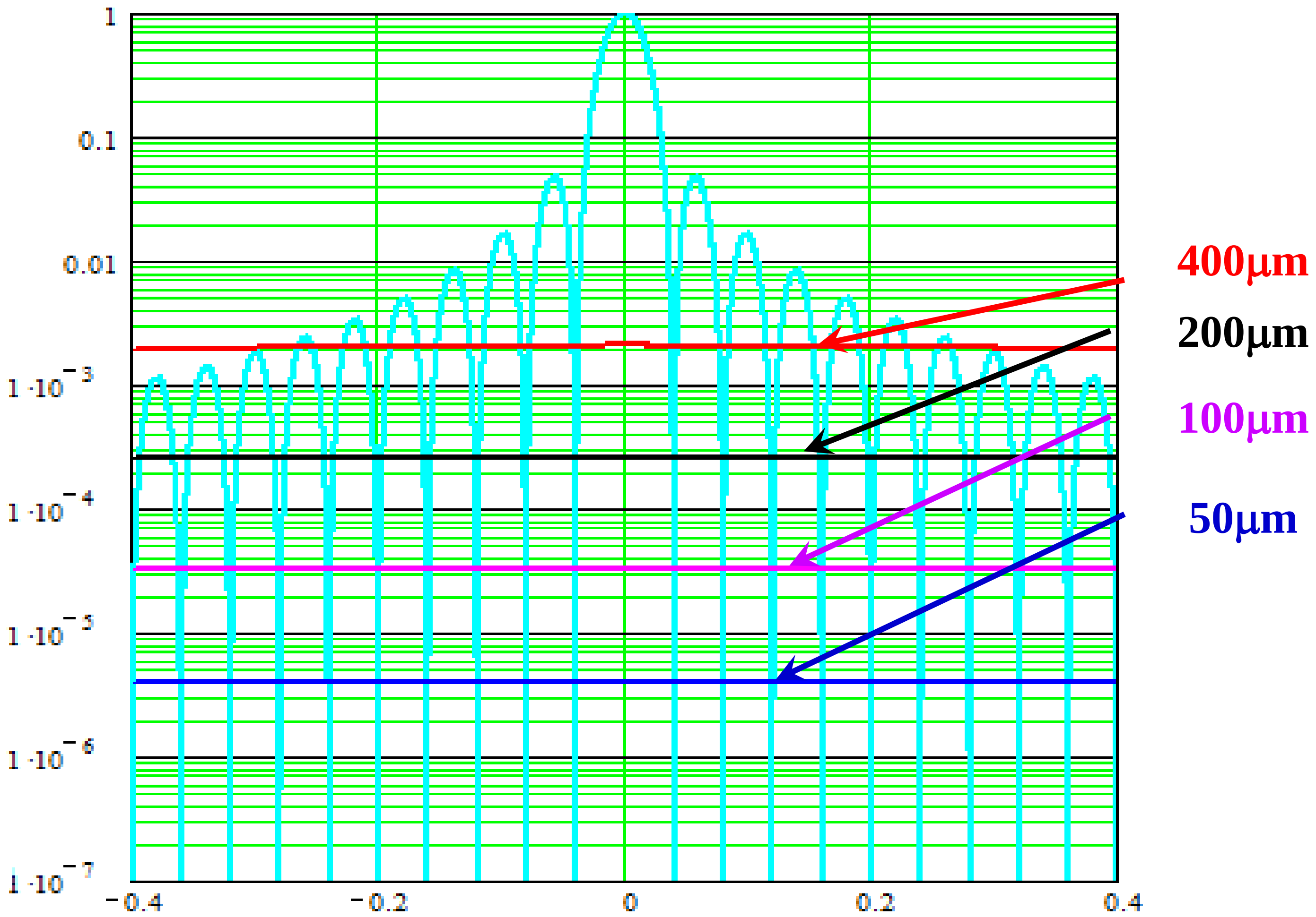
To escape from noise produced by the objective lens is most important issue in the coronagraph!!

Simulation result of Background produced by dig on objective surface





1
0.1
0.01
1·10⁻³
1·10⁻⁴
1·10⁻⁵
1·10⁻⁶
1·10⁻⁷
−10
−5
0
5
10
400μm
200μm
100μm
50μm
Diffraction by objective lens aperture



1
0.1
0.01
1·10⁻³
1·10⁻⁴
1·10⁻⁵
1·10⁻⁶
1·10⁻⁷
−0.4
−0.2
0
0.2
0.4
400µm
200µm
100µm
50µm

How to eliminate Mie scattering??

1. A careful optical polishing for the objective lens.
2. Reduce number of glass surface.

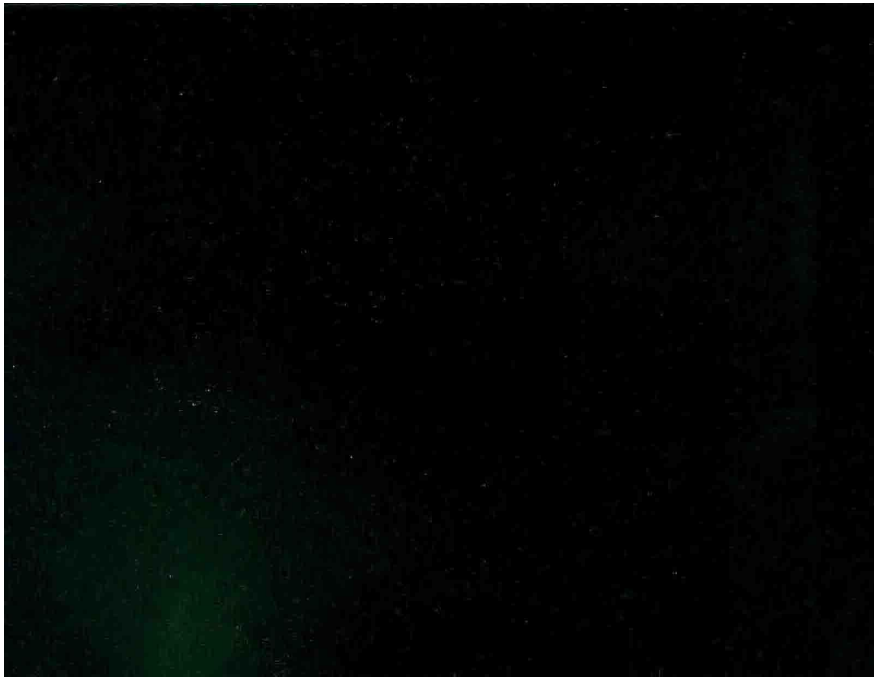
   use a singlet lens for the objective lens.
3. No coating (Anti-reflection, Neutral density etc.) for objective

A careful optical polishing for the objective lens

5mm

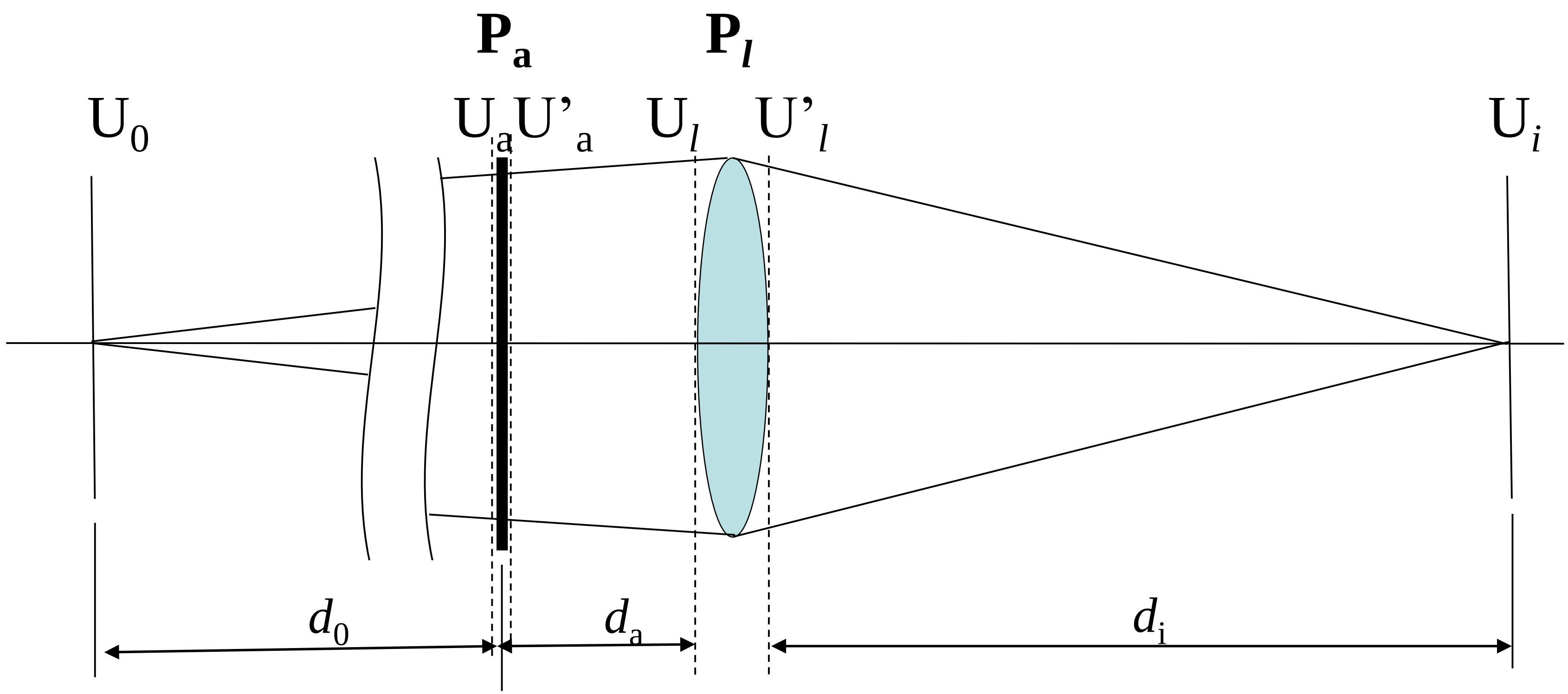
Comparison between normal optical polish and careful optical polish for coronagraph

S&D 60/40 surface of the lens

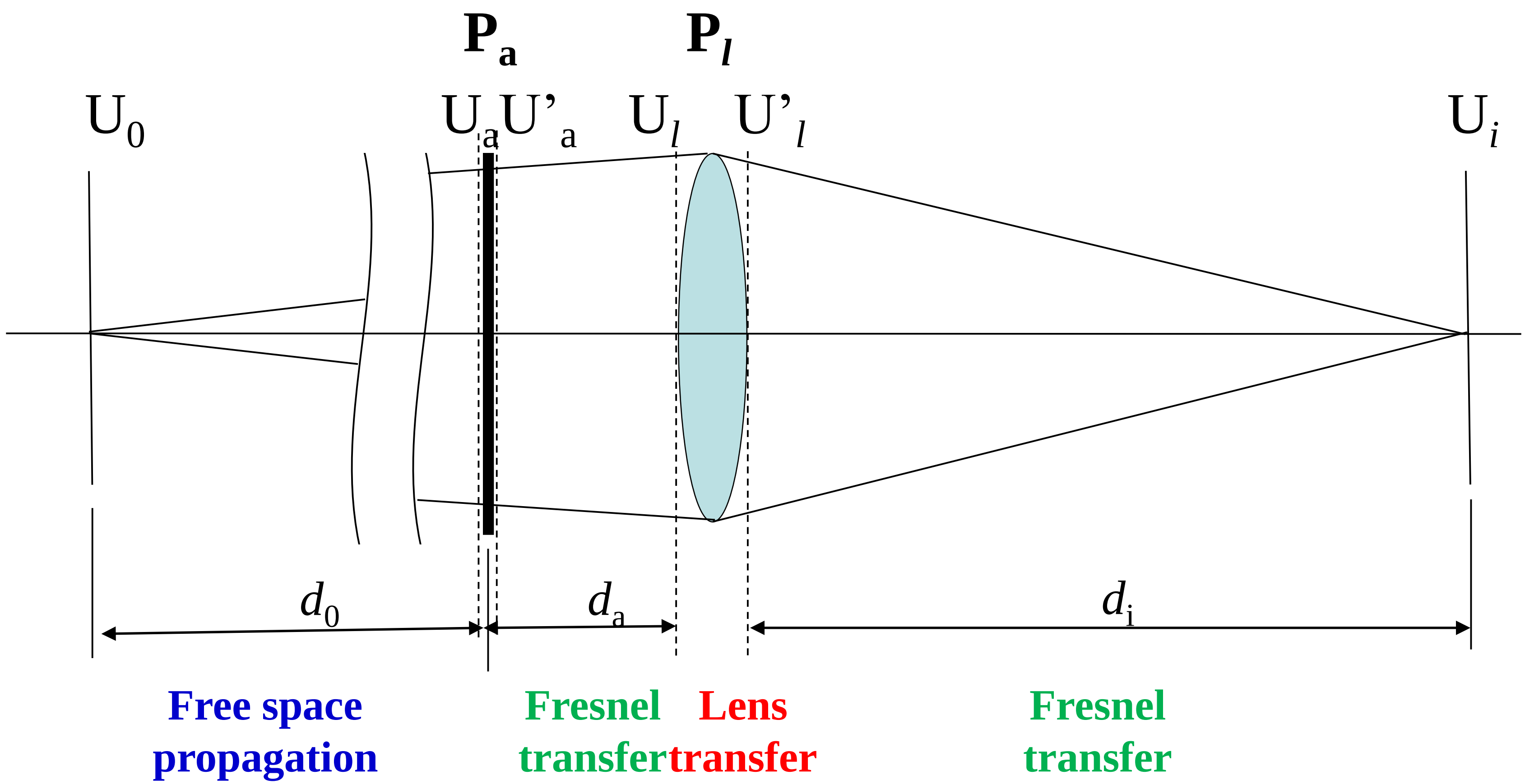


Surface of the coronagraph lens

Case 2. Noise source in front of the objective lens



Noise source in front of the lens



After tired calculations,

$$U_i(x_i, y_i) = \iint \left[\iint P_a(x_a, y_a) exp\left\{ i\frac{k}{2}\left(\frac{1}{d_0 - d_a} - \frac{1}{d_a} \right)\left(x_a{}^2 + y_a{}^2 \right) \right\} \right.$$

$$\left. \cdot exp\left\{ -ik\left(\left(\frac{x_0}{d_0 - d_a} + \frac{x_l}{d_a} \right) x_a + \left(\frac{y_0}{d_0 - d_a} + \frac{y_l}{d_a} \right) y_a \right) \right\} dx_a dy_a \right]$$

$$\cdot P_l(x_l, y_l) exp\left\{ i\frac{k}{2}\left(\frac{1}{d_l} + \frac{1}{d_i} - \frac{1}{f} \right)\left(x_l{}^2 + y_l{}^2 \right) \right\}$$

$$\cdot exp\left\{ -i\frac{k}{d_i}\left(x_i\ x_l + y_i\ y_l \right) \right\} dx_l dy_l$$

$$in\, here,\ d_l = d_a + d_0$$

After tired calculations,

Diffraction by noise source

$$U_i(x_i, y_i) = \iint \Bigg[\iint P_a(x_a, y_a) \exp\left\{ i \frac{k}{2} \left(\frac{1}{d_0 - d_a} - \frac{1}{d_a} \right) \left(x_a{}^2 + y_a{}^2 \right) \right\}$$

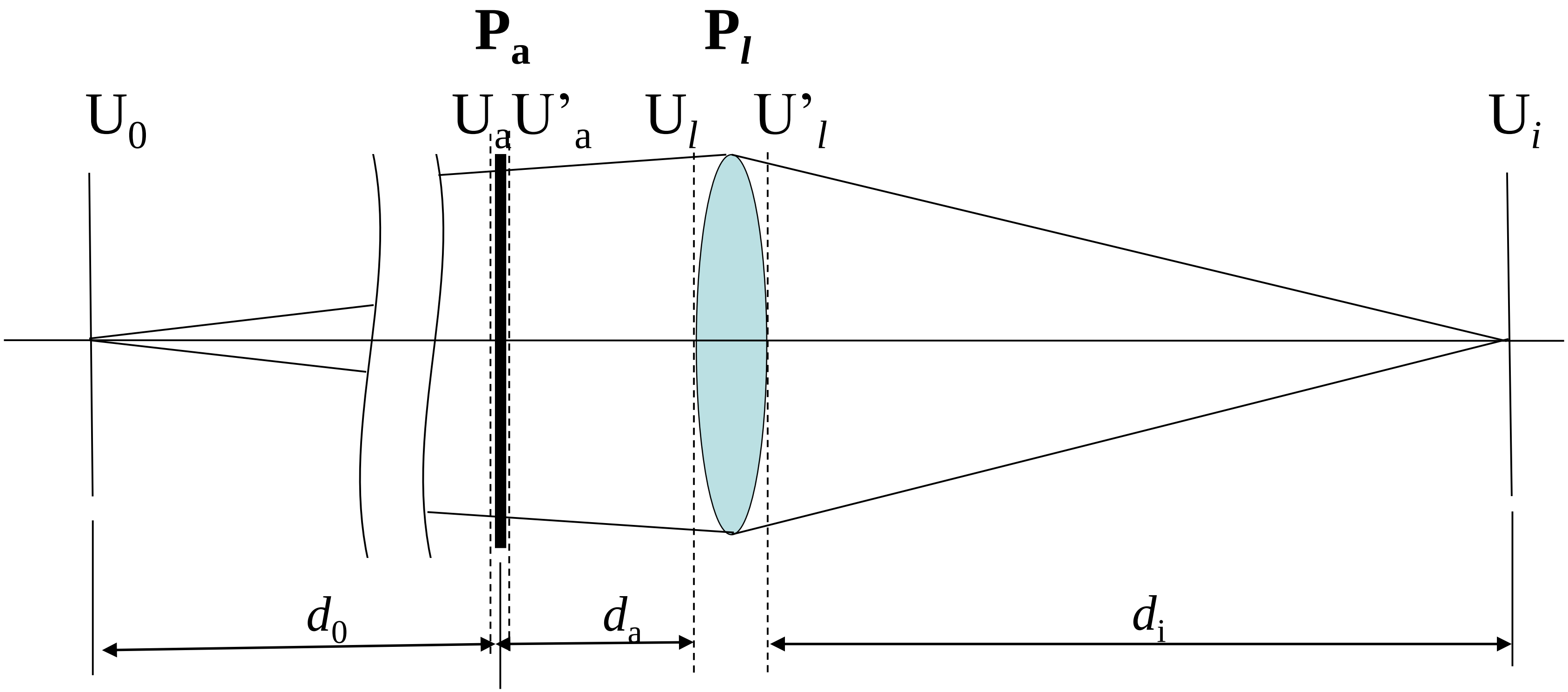
$$\cdot \exp\left\{ -ik \left(\left(\frac{x_0}{d_0 - d_a} + \frac{x_l}{d_a} \right) x_a + \left(\frac{y_0}{d_0 - d_a} + \frac{y_l}{d_a} \right) y_a \right) \right\} dx_a dy_a \Bigg]$$

$$\cdot P_l(x_l, y_l) \exp\left\{ i \frac{k}{2} \left(\frac{1}{d_l} + \frac{1}{d_i} - \frac{1}{f} \right) \left(x_l{}^2 + y_l{}^2 \right) \right\}$$

$$\cdot \exp\left\{ -i \frac{k}{d_i} \left(x_i\, x_l + y_i\, y_l \right) \right\} dx_l dy_l$$

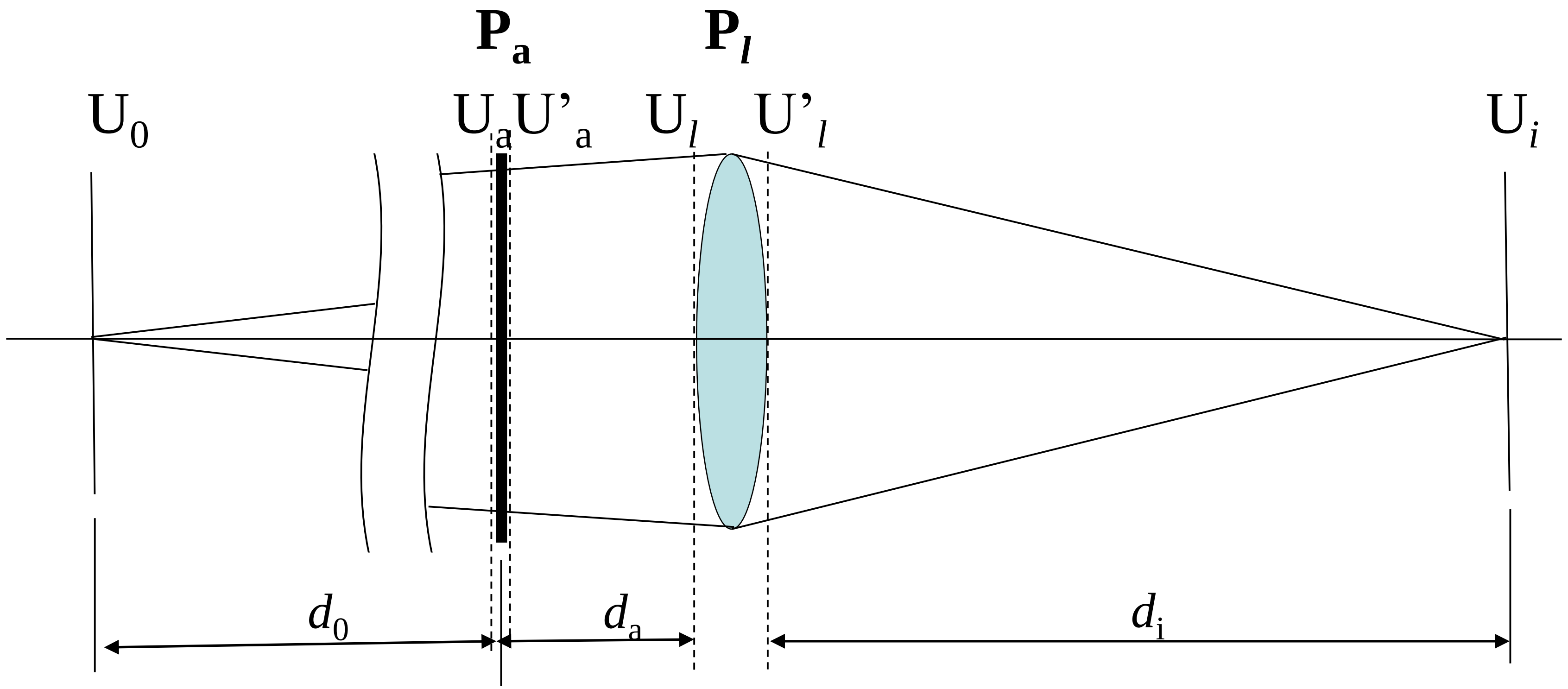
Diffraction by lens pupil

Noise source in front of the lens



Noise source ⟹ Fresnel diffraction

Then re-diffracted by lens pupil



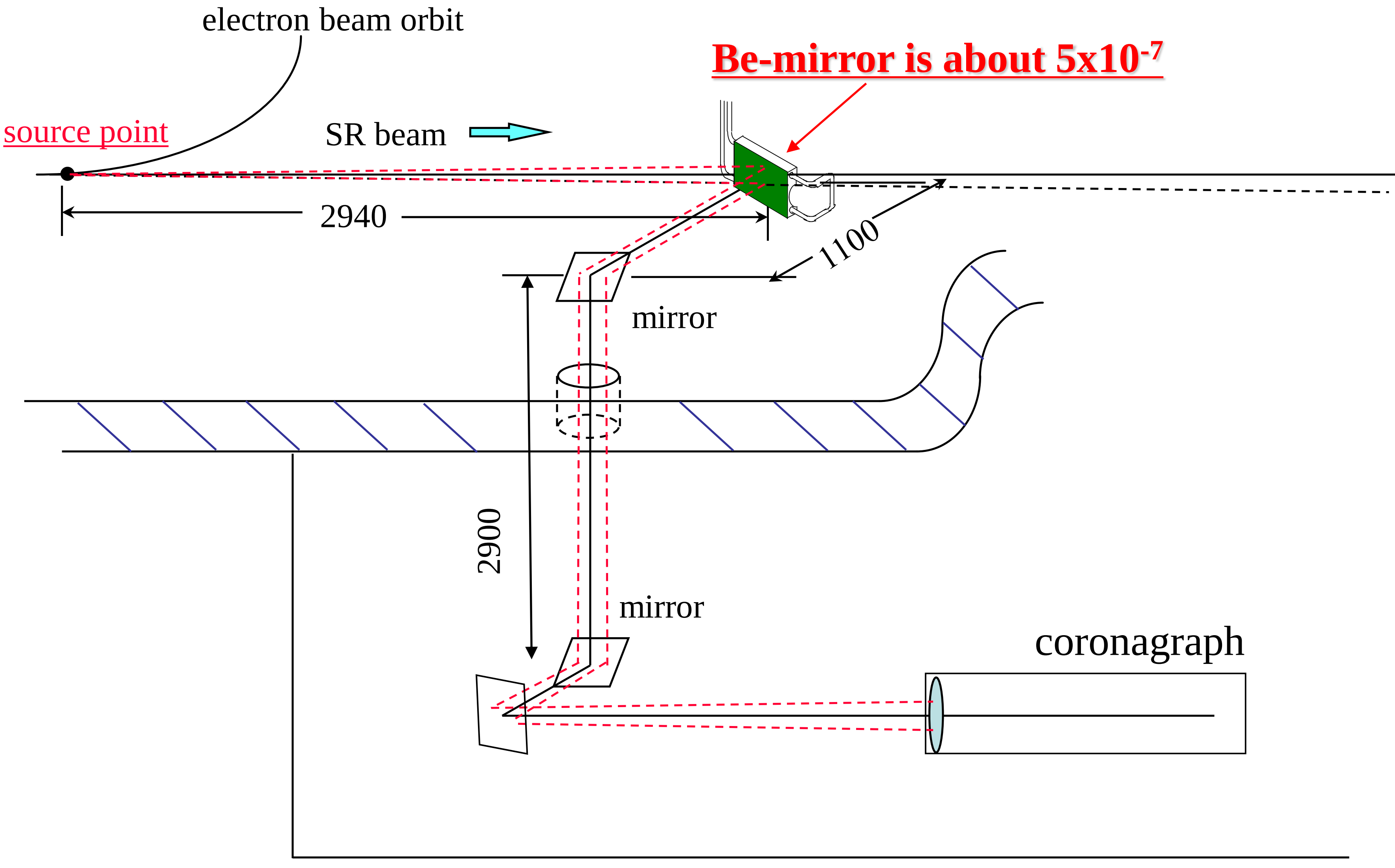
Noise source to objective lens ⇨ Fresnel like diffraction

Then this input is re-diffracted by objective lens pupil

d_a is shorter : out of focus image of noise source +Fresnel like diffraction

d_a is longer : quasi-focused image of noise source +Fraunhofer like diffraction

Set up of SR monitor at the Photon factory



Scratch and dig on extraction mirror at ASZL

Veeco

Mag: 2.5 X

Mode: PSI

Surface Data

Date: 03/09/2006

Time: 17:57:50

Surface Statistics:

Ra: 5.14 A

Rq: 6.54 A

Rz: 58.42 A

Rt: 67.41 A

Rad Crv: -1.09 km

Set-up Parameters:

Size: 736 X 480

Sampling: 3.31 um

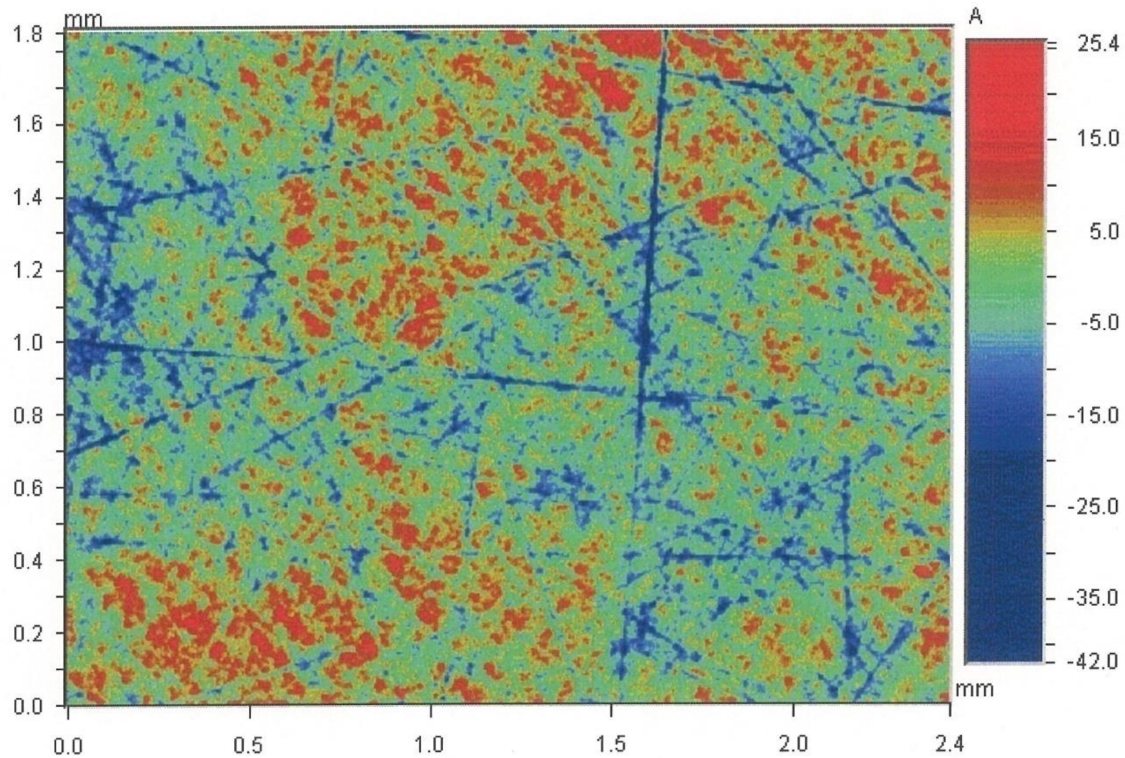
Processed Options:

Terms Removed:

Curvature & Tilt

Filtering:

None



Title: No3

Note: 3Dx2.5

Low noise mirror for optical path

To escape from noise from mirrors in the optical path, we used back-coated mirror with well polished optical flat having a small wedge angle.

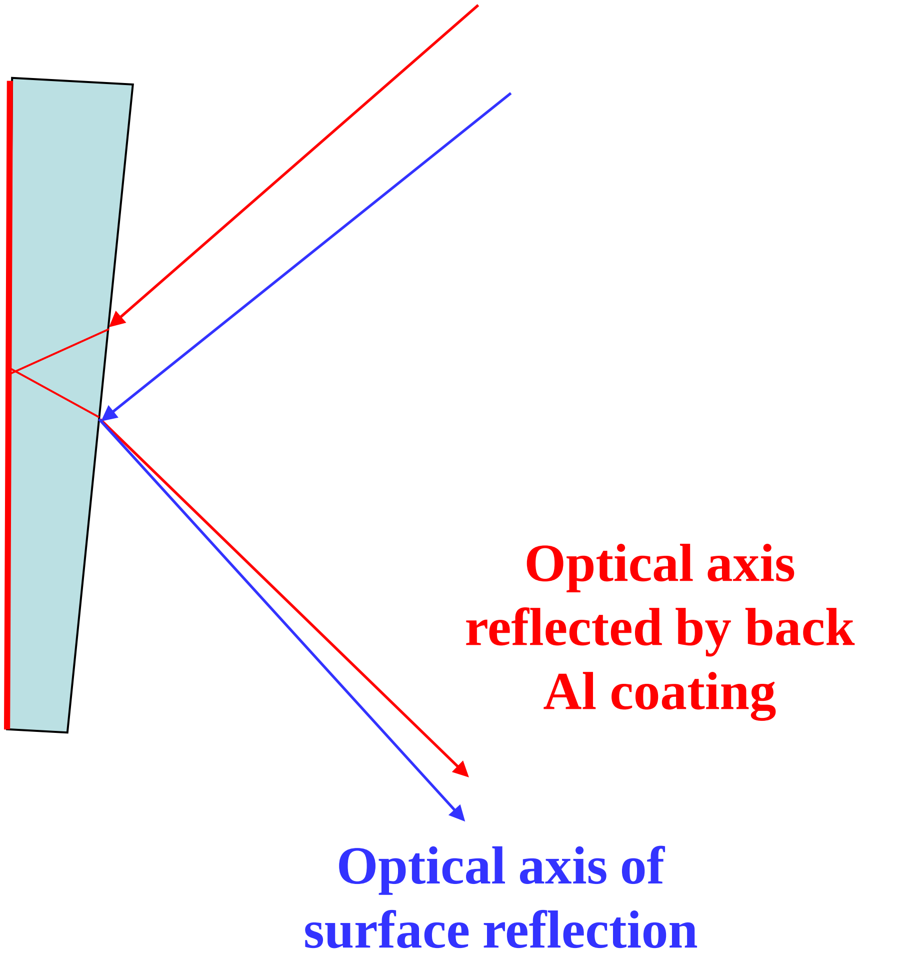


Image of beam profile without the opaque disk. Exposure time of CCD camera is 10msec.

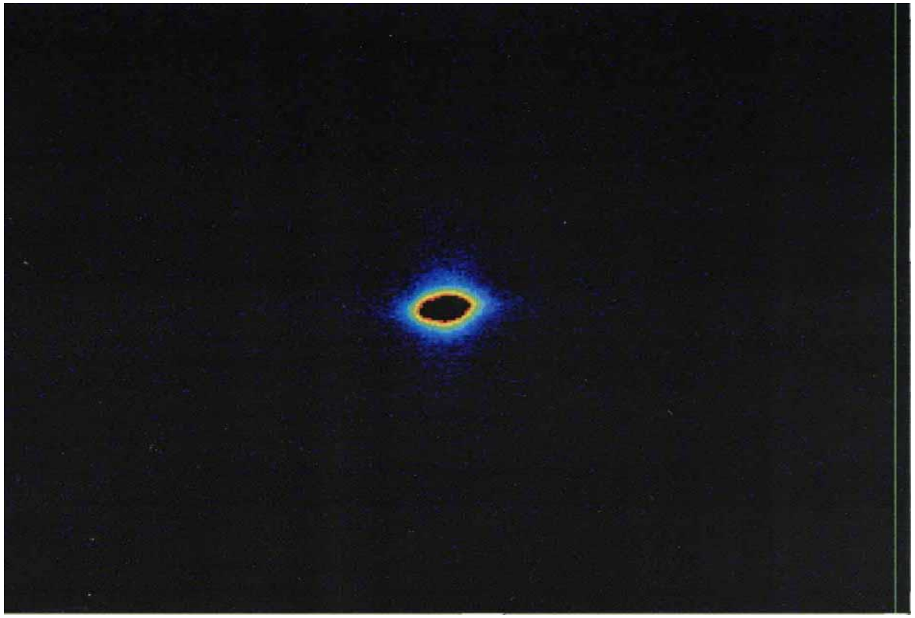
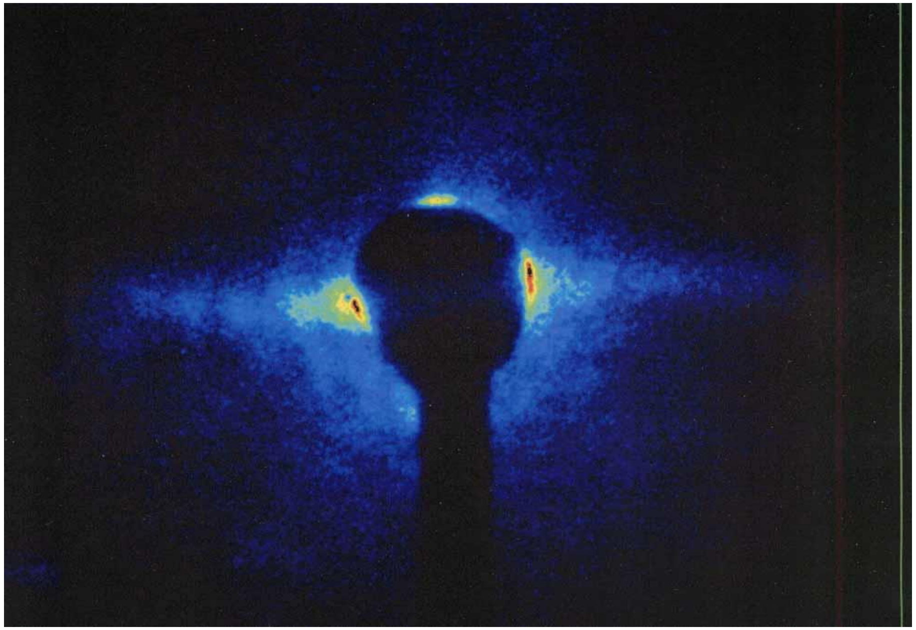
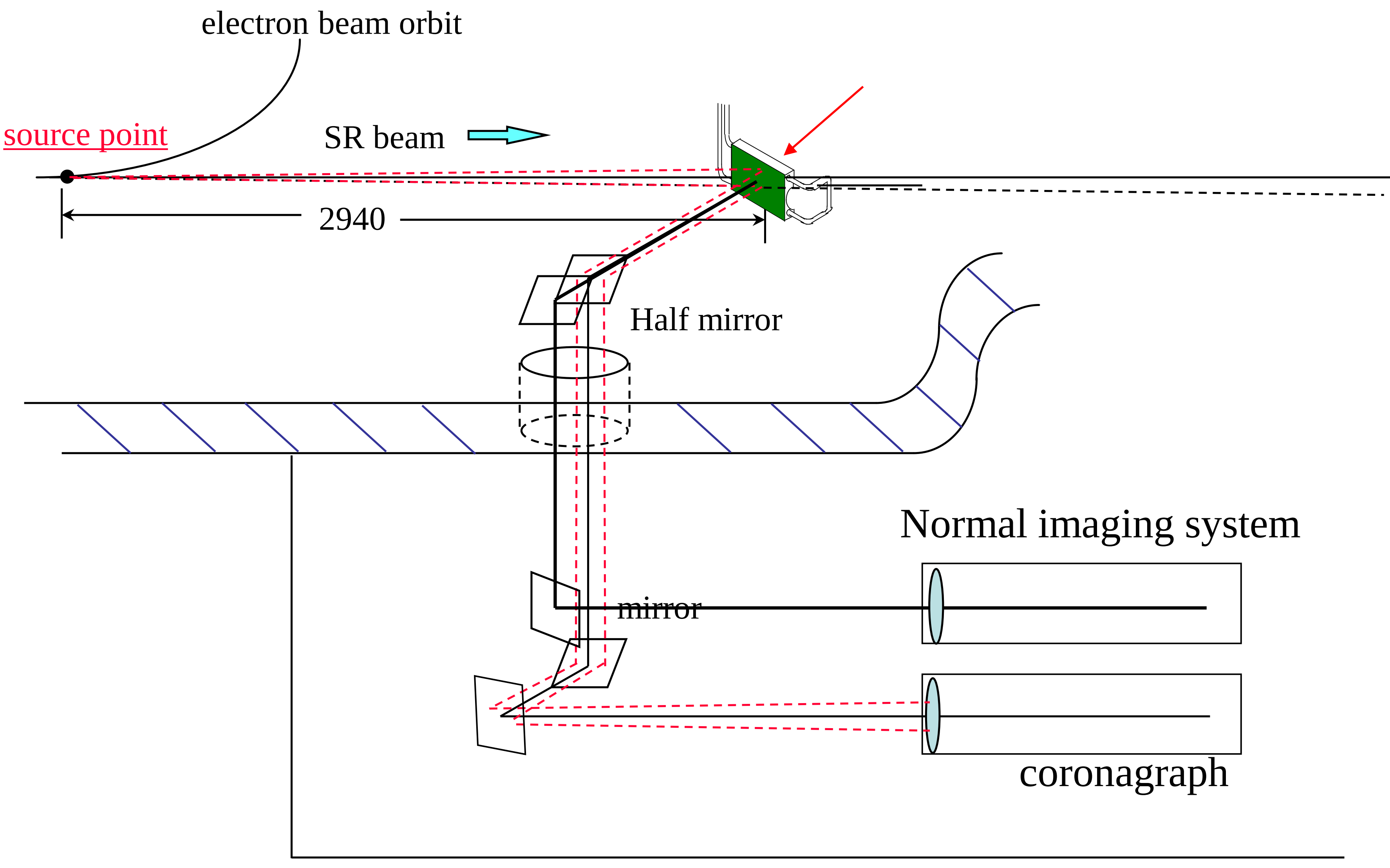


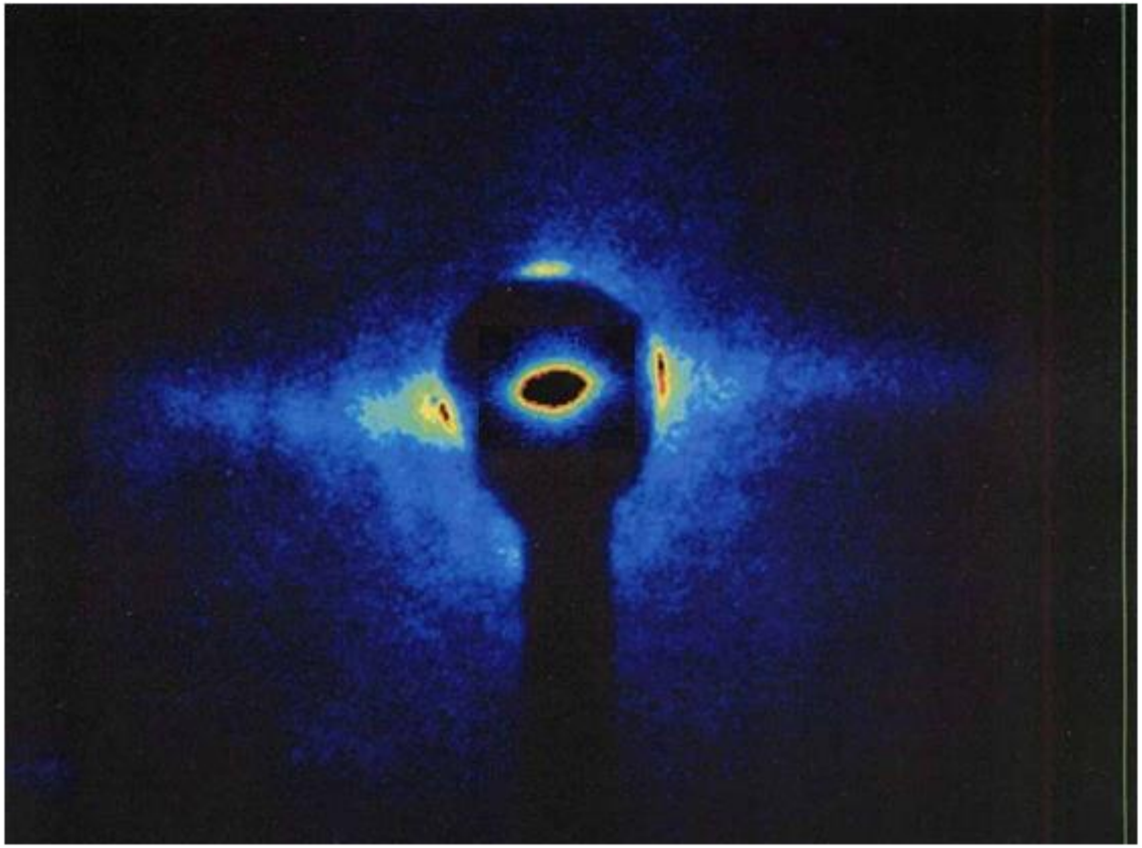
Image of beam tail with the opaque disk. Transverse magnification is same as in Fig.6. Exposure time of CCD camera is 10msec.



Beam core + halo (superimposed)

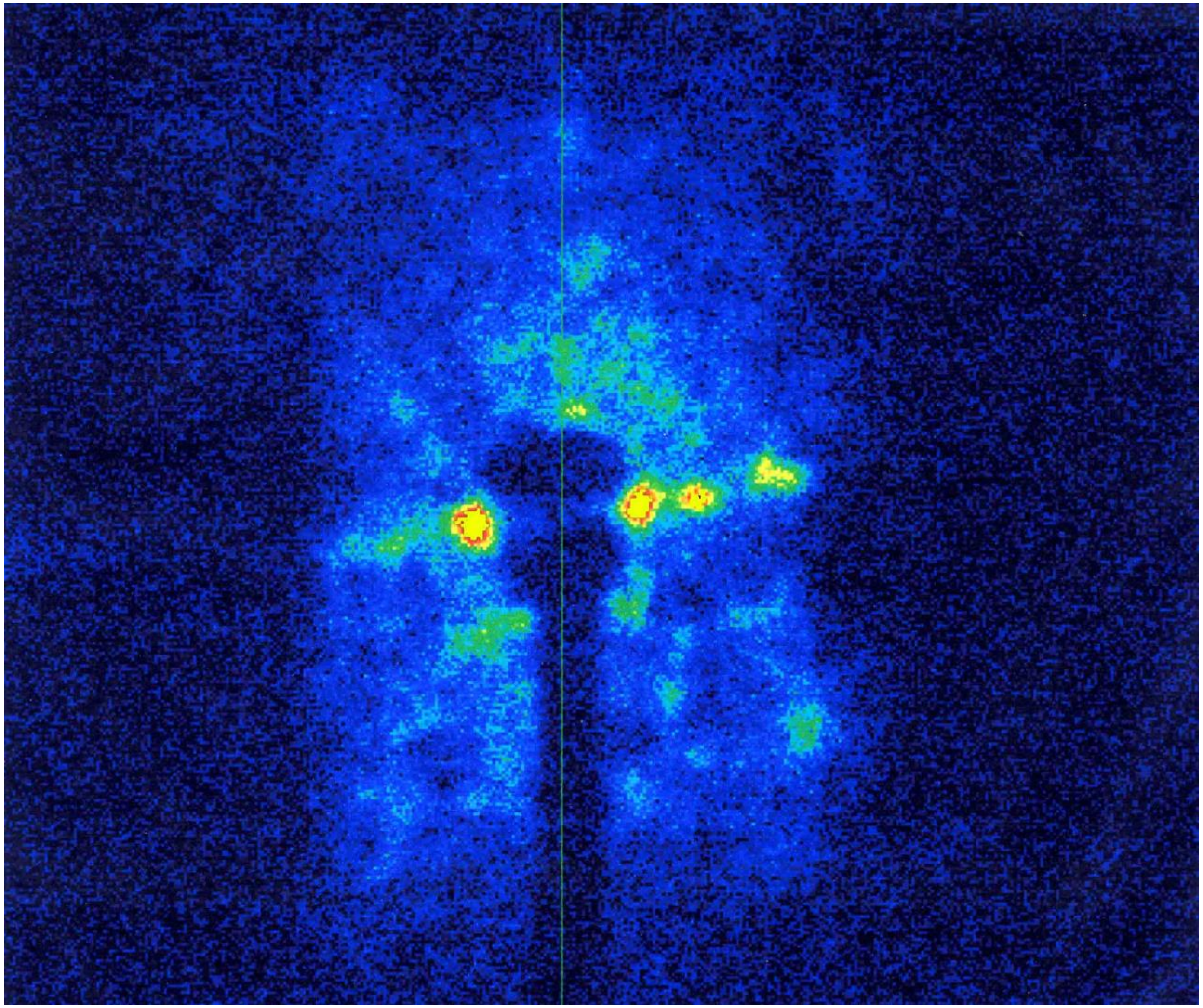


Beam core + halo (superimposed)



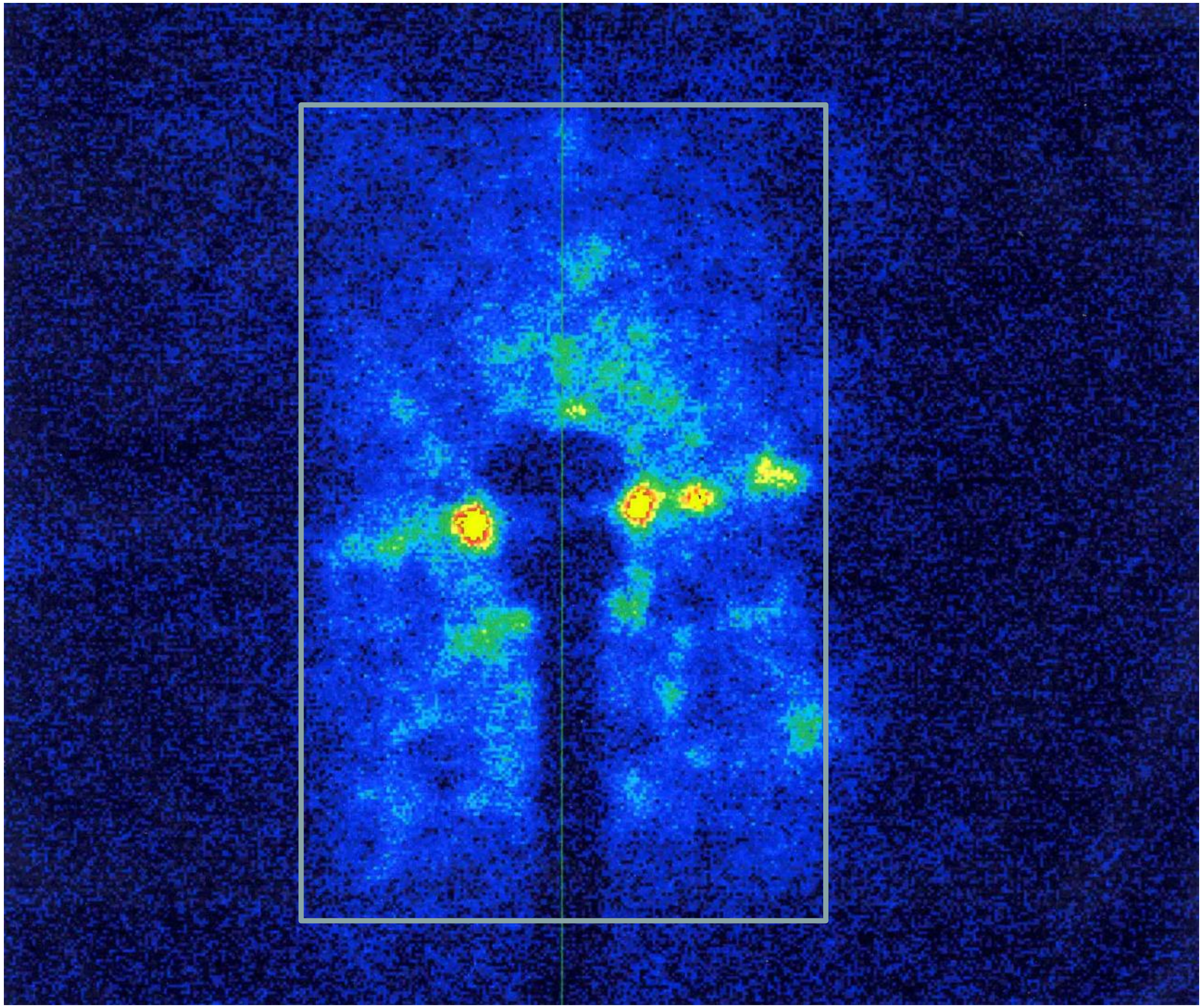
Effect of Mie scattering noise

Intentionally spread some dust on the mirror in 2m front of the coronagraph

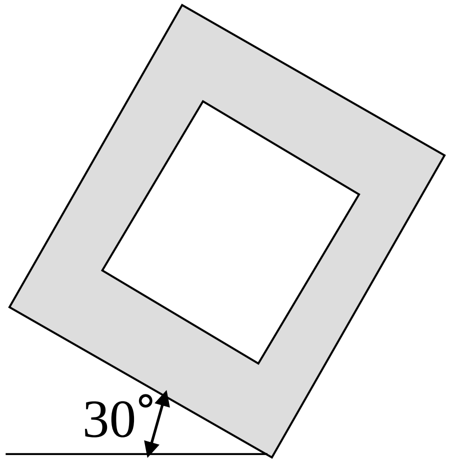


Effect of Mie scattering noise

Intentionally spread some dust on the mirror in 2m front of the coronagraph

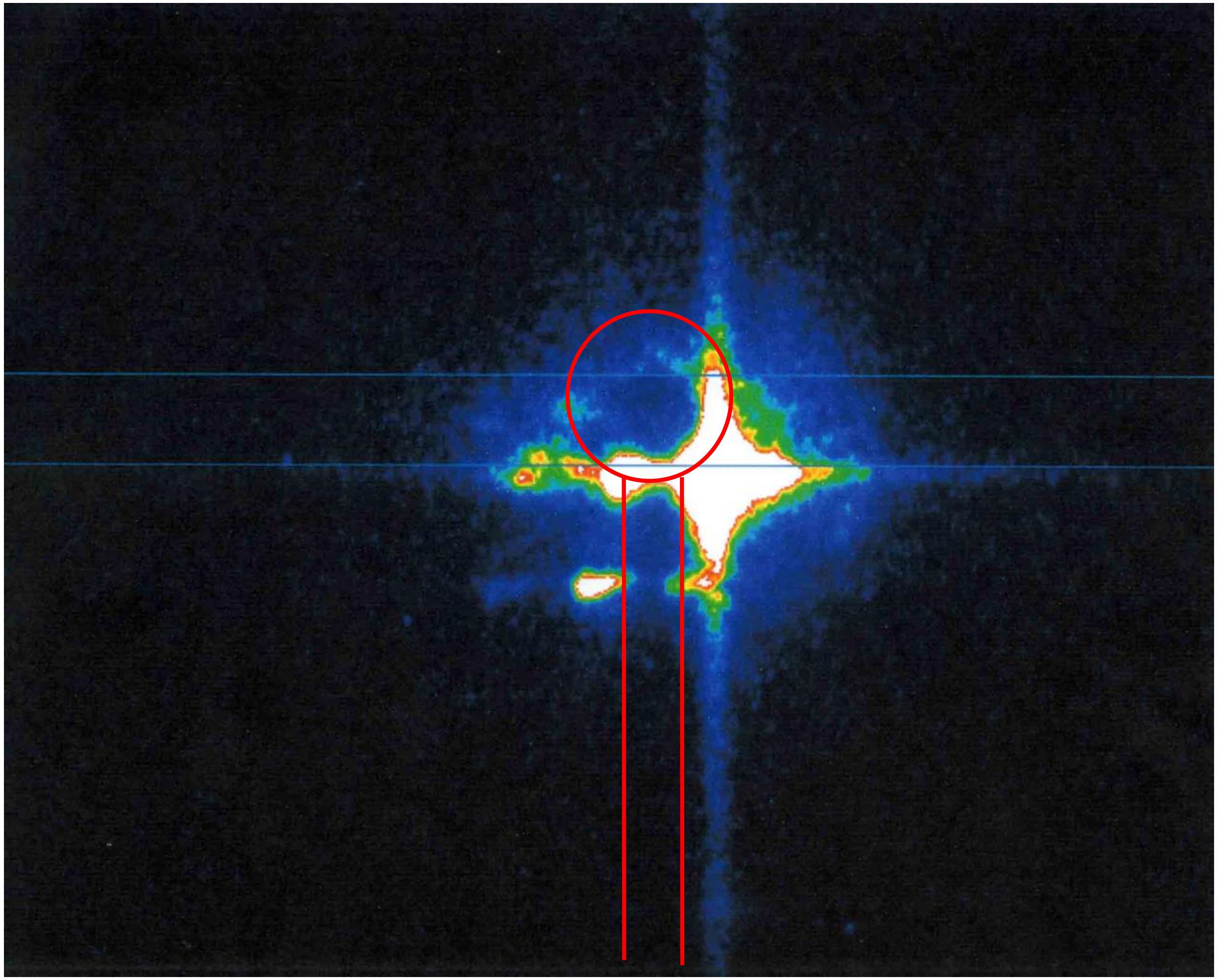


Diffraction tail observed without Lyot's stop

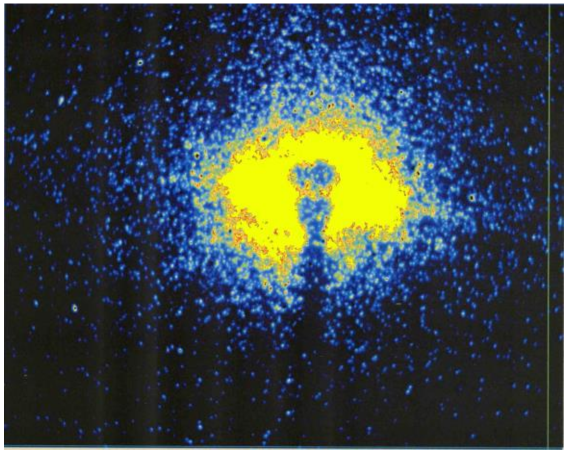


Entrance pupil is intentionally rotated by 30º to recognize diffraction tail easily.

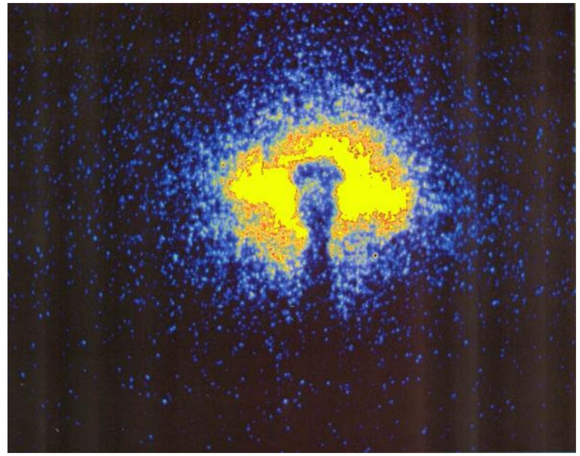
Move the opaque disk slightly to show the edge of central beam image (diamond ring!)



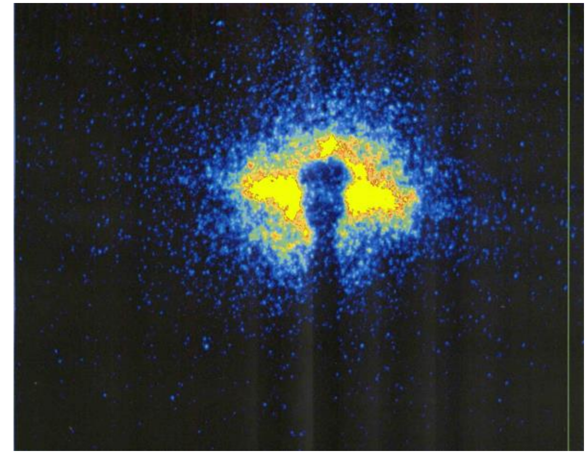
Beam tail images in the single bunch operation at the KEK PF measured at different current



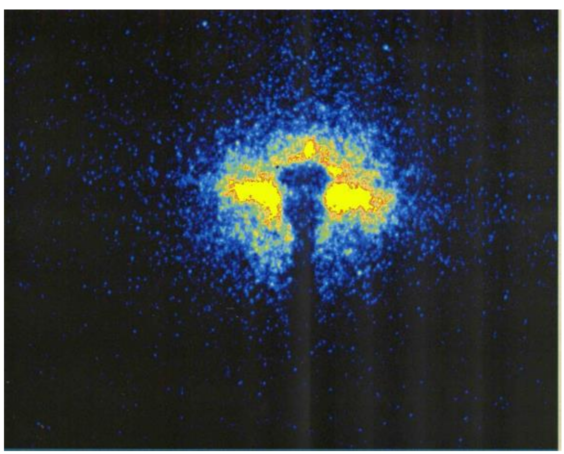
65.8mA



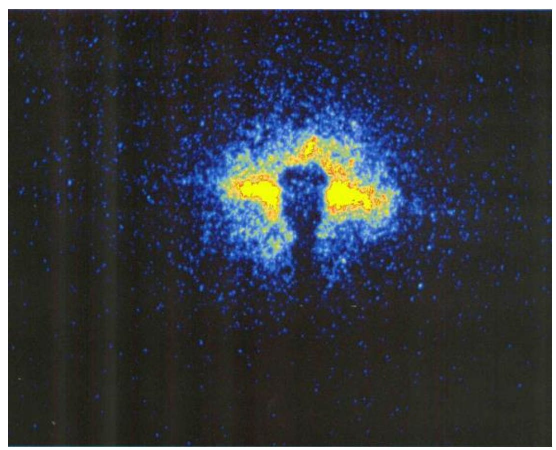
61.4mA



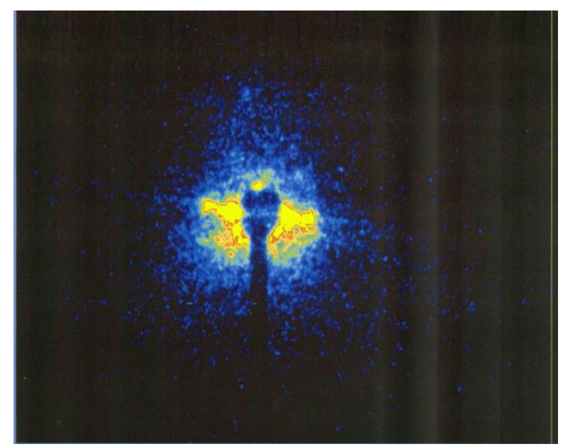
54.3mA



45.5mA



35.5mA

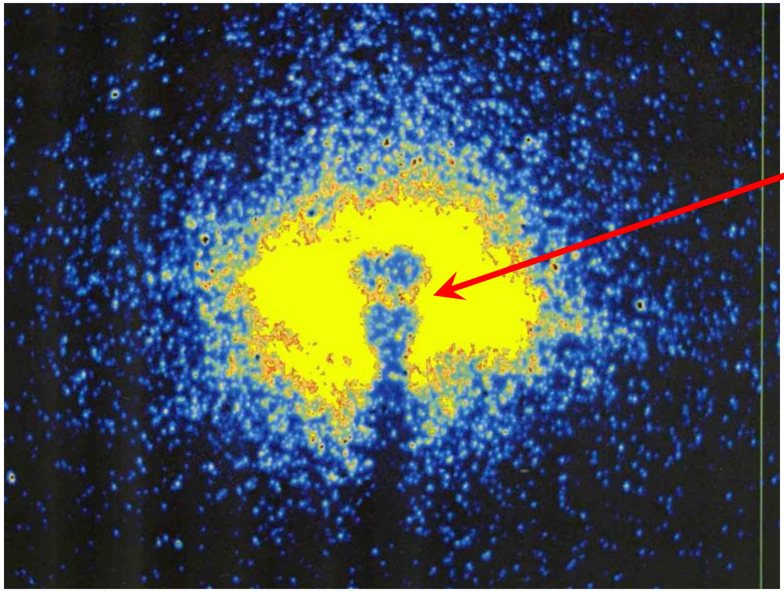


396.8mA
Multi-bunch
bunch current 1.42mA

Observation for the deep out side

Single bunch 65.8mA

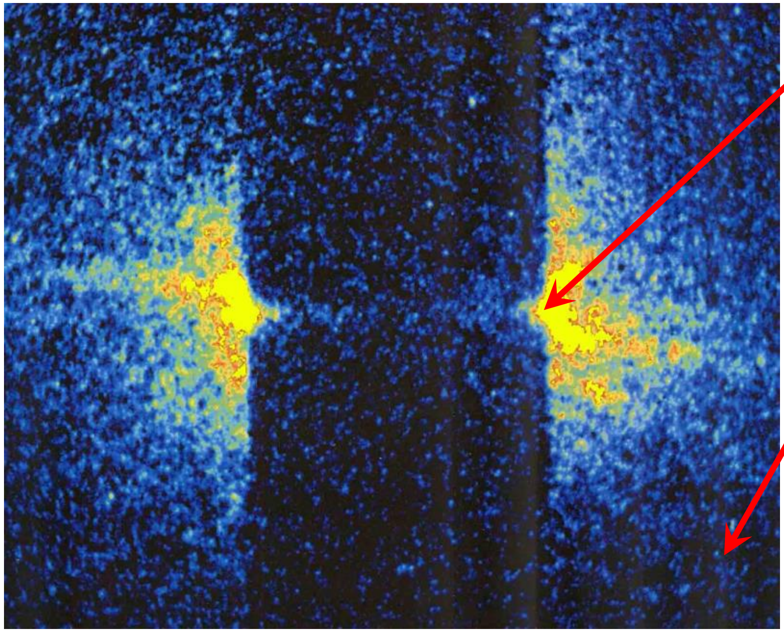
Exposure time of CCD : 3msec



Intensity in here : $2.05\text{x}10^{-4}$ of peak intensity

Deep outside

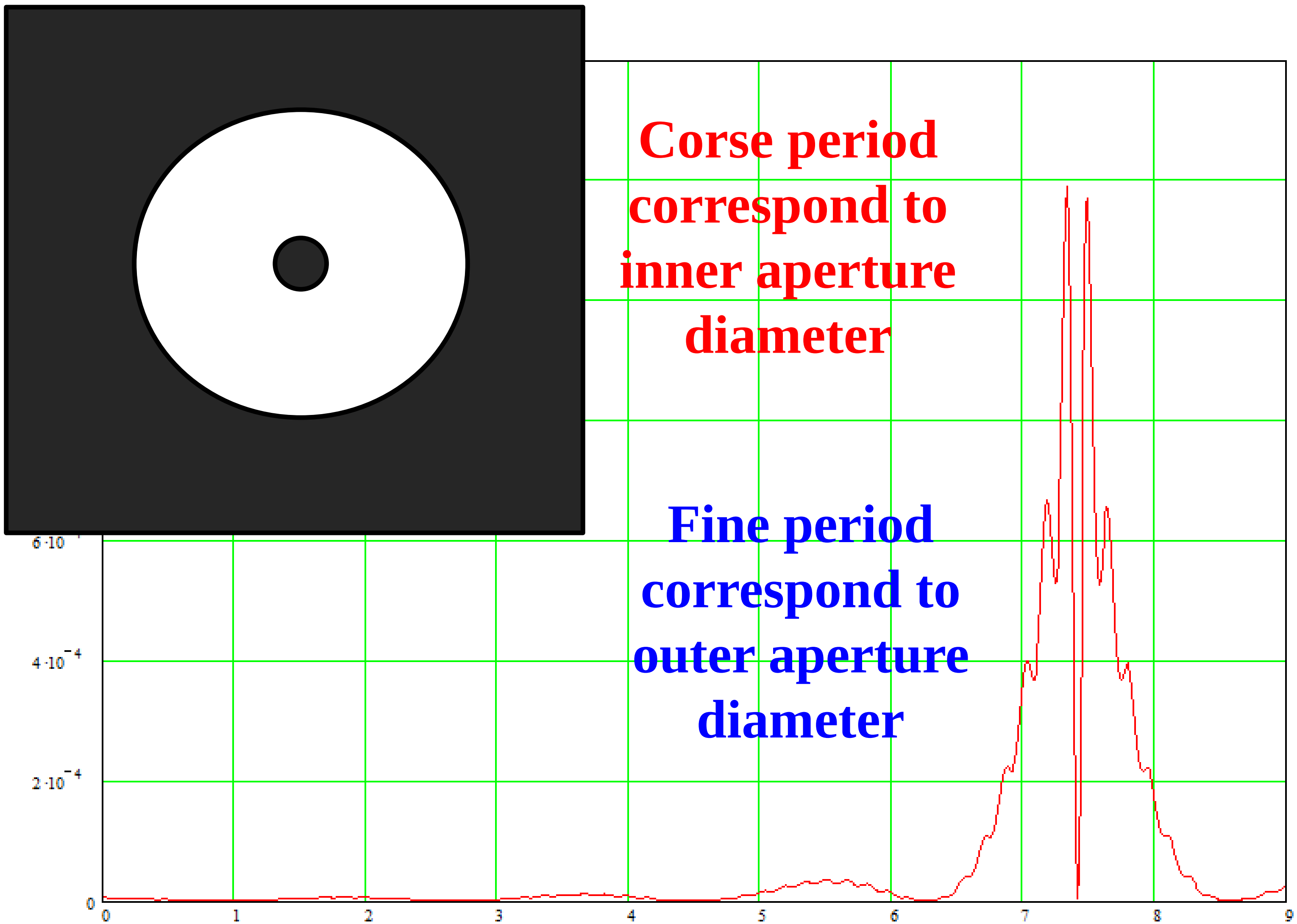
Exposure time of CCD : 100msec



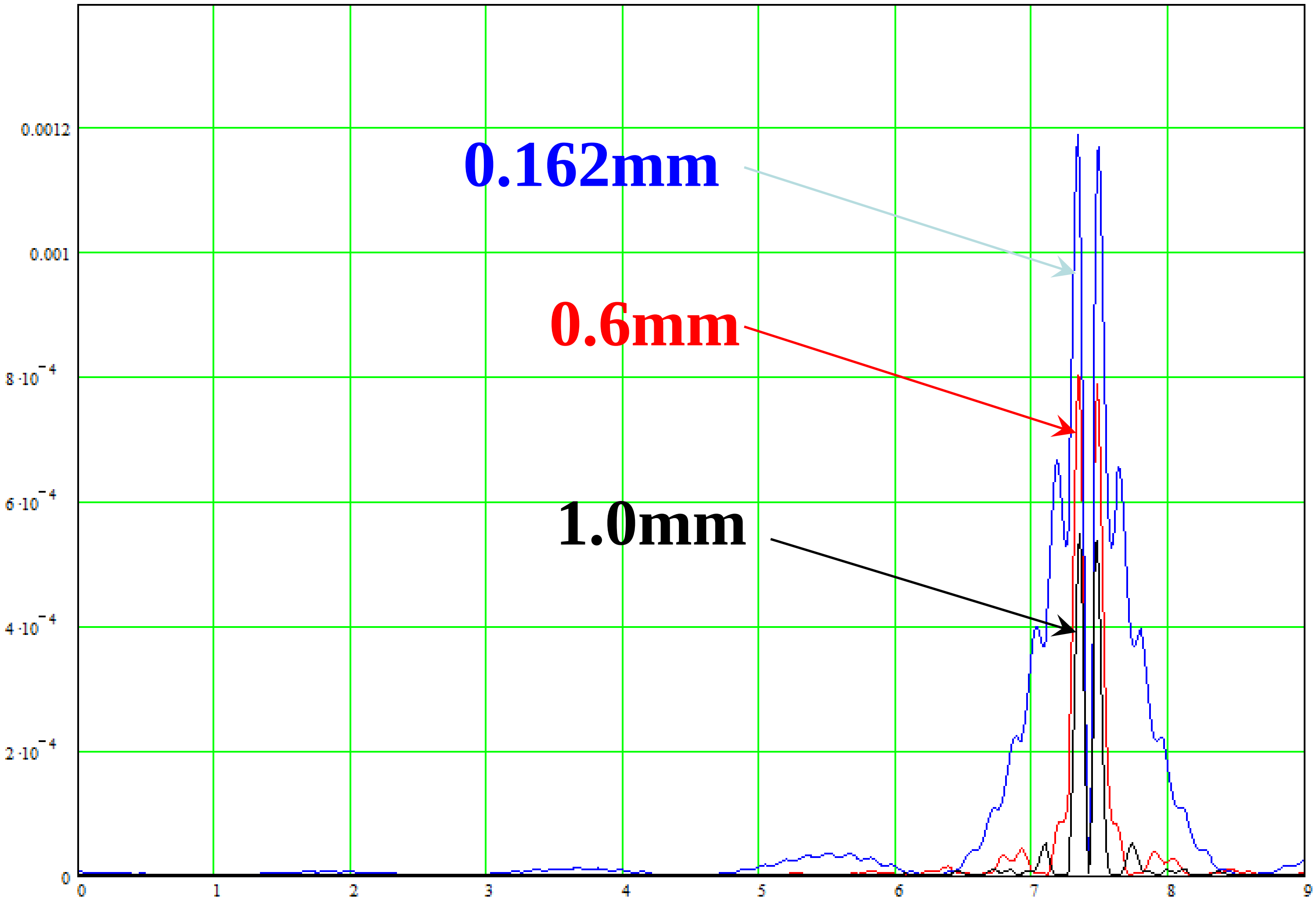
$2.55\text{x}10^{-6}$

Background leavel : about $6\text{x}10^{-7}$

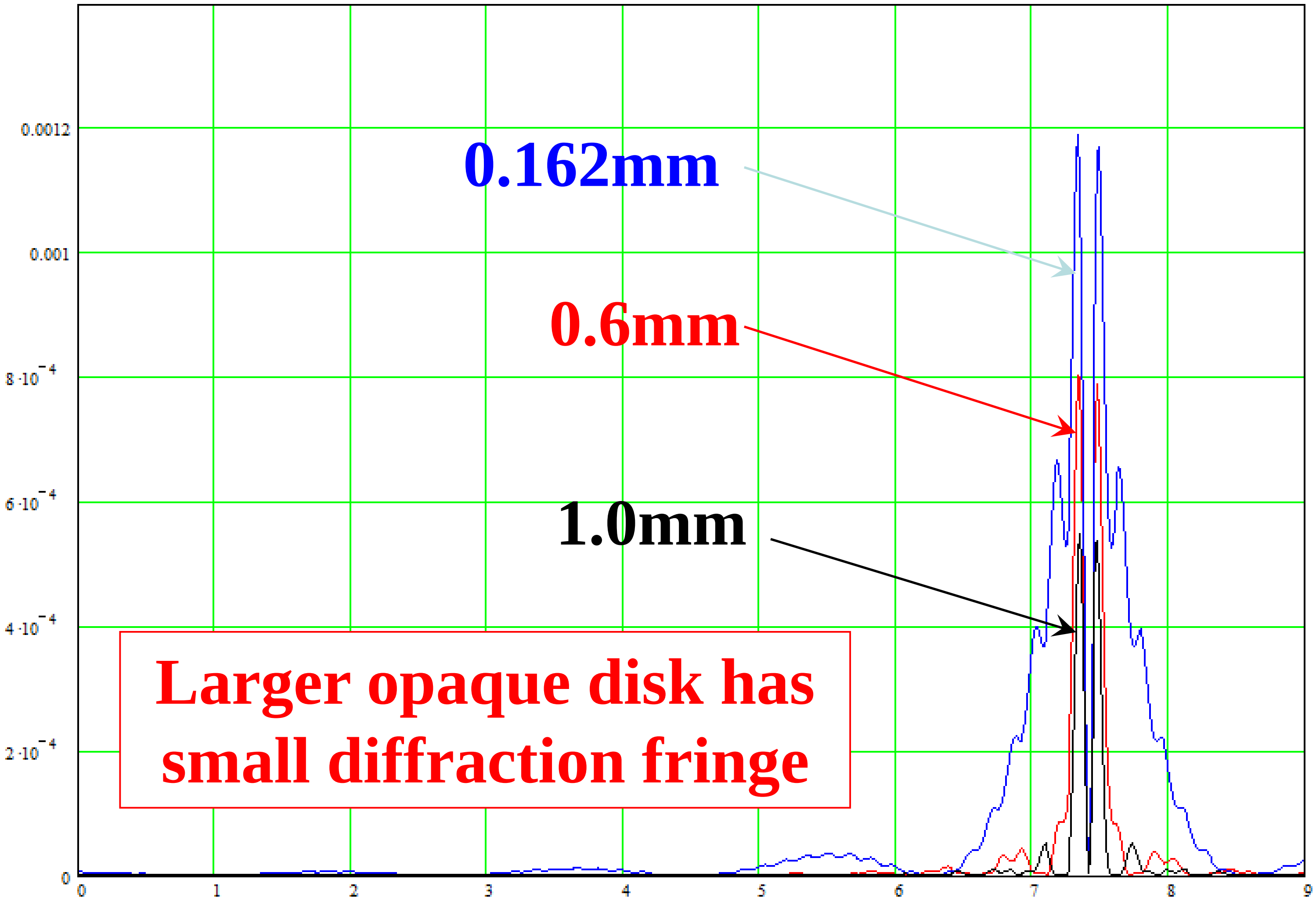
How is the performance of coronagraph limited?


Corse period correspond to inner aperture diameter
Fine period correspond to outer aperture diameter
6·10
4·10−4
2·10−4
0
0
1
2
3
4
5
6
7
8
9

Back to diffraction fringe on Lyot stop



Back to diffraction fringe on Lyot stop



Phase 2 coronagraph for HL LHC

1. **The leakage of diffraction fringe reduces by increase of opaque disk diameter.**

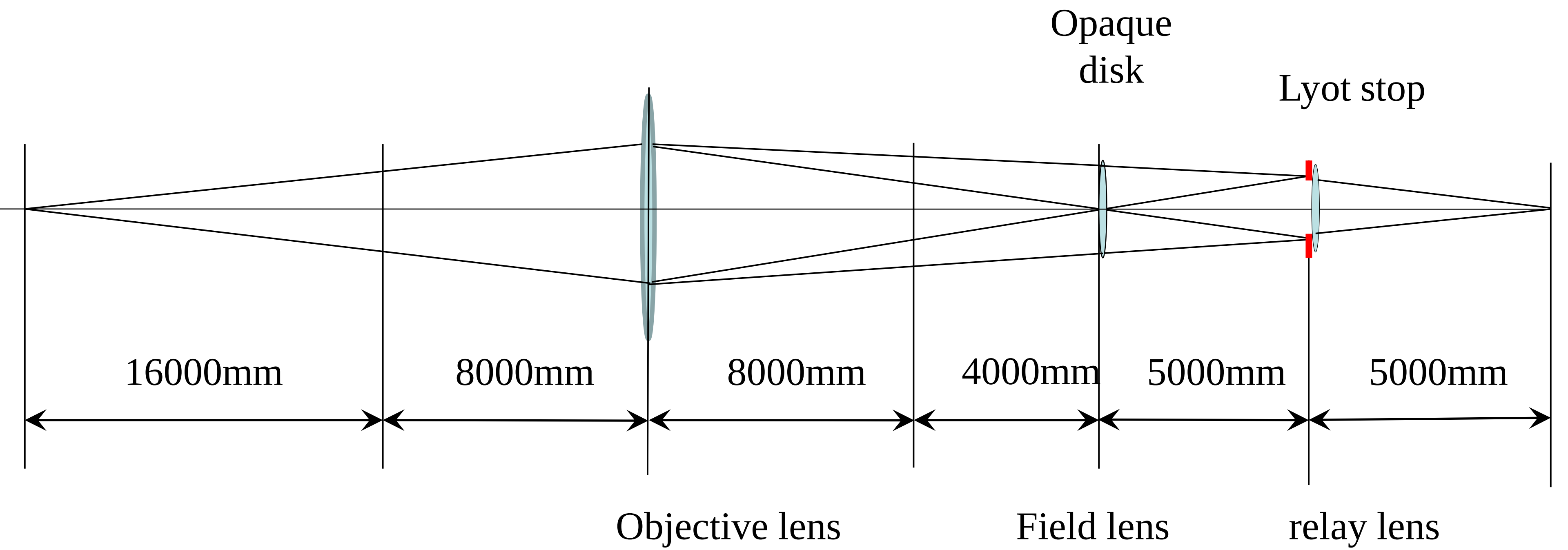
We can realize better contrast with larger transverse magnification.

We should design the focal length of objective lens longer.

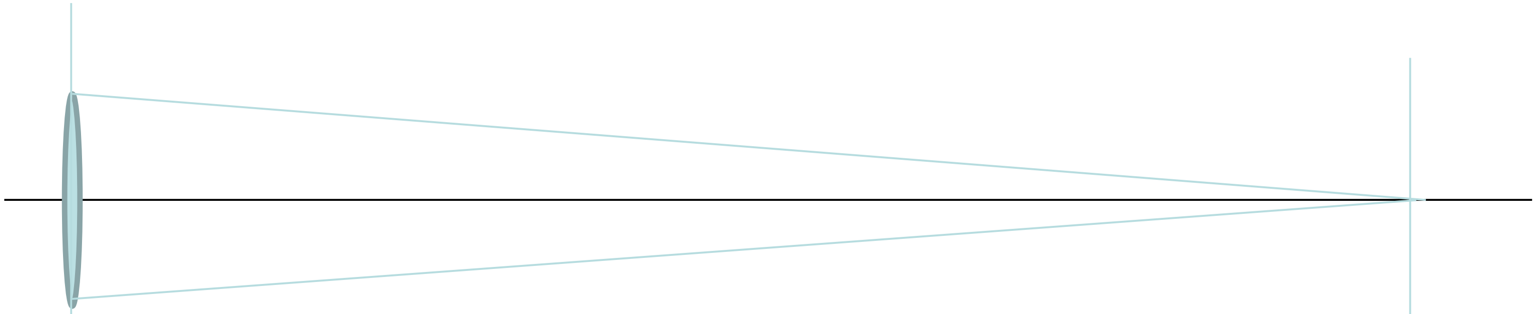
2. **Chromatic shift on optical axis is given by focal length divided by the Abbe number, designing long focal length increases drastically increase the chromatic foal shift.**

We should apply reflector focusing system for the objective instead of the objective lens system in phase 2.

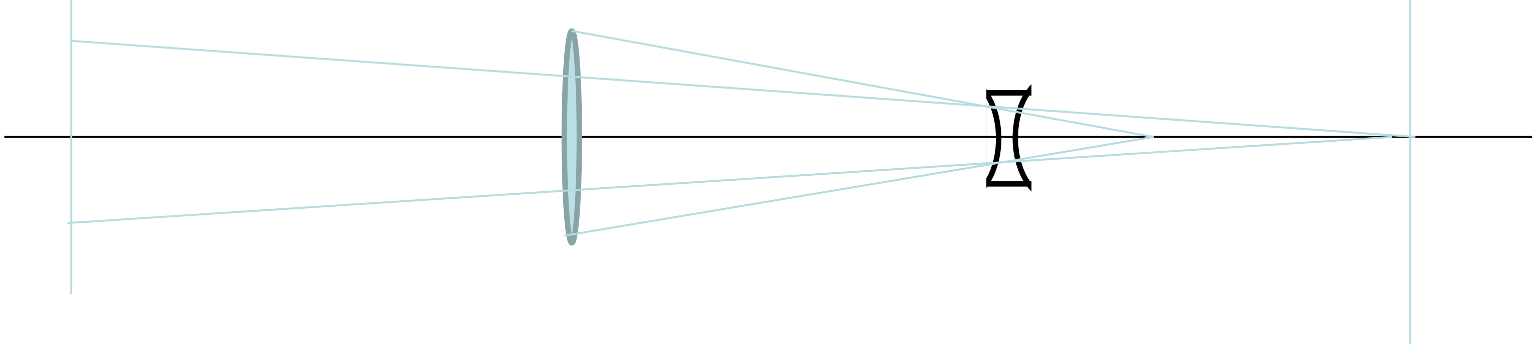
Coronagraph having a magnification of 0.5 (about 7 times larger transverse magnification)



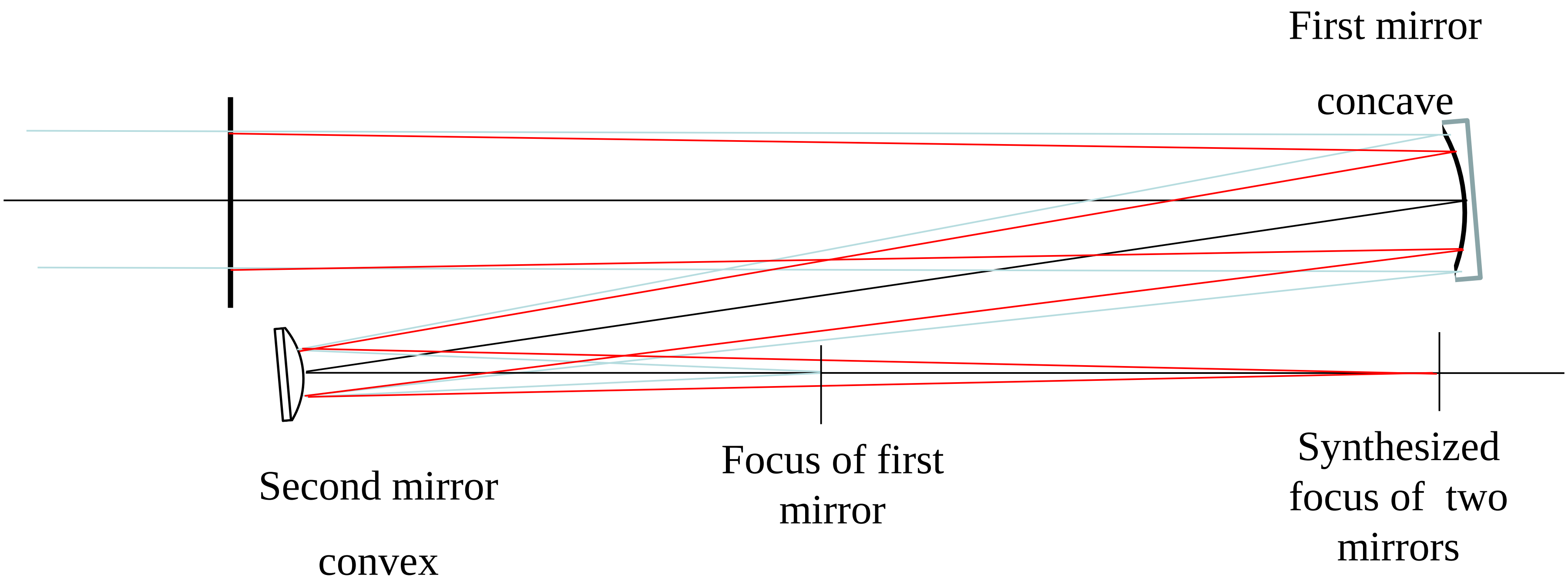
Reduction the actual length of long focus of the simple objective lens with a combination of focus and defocus lenses.



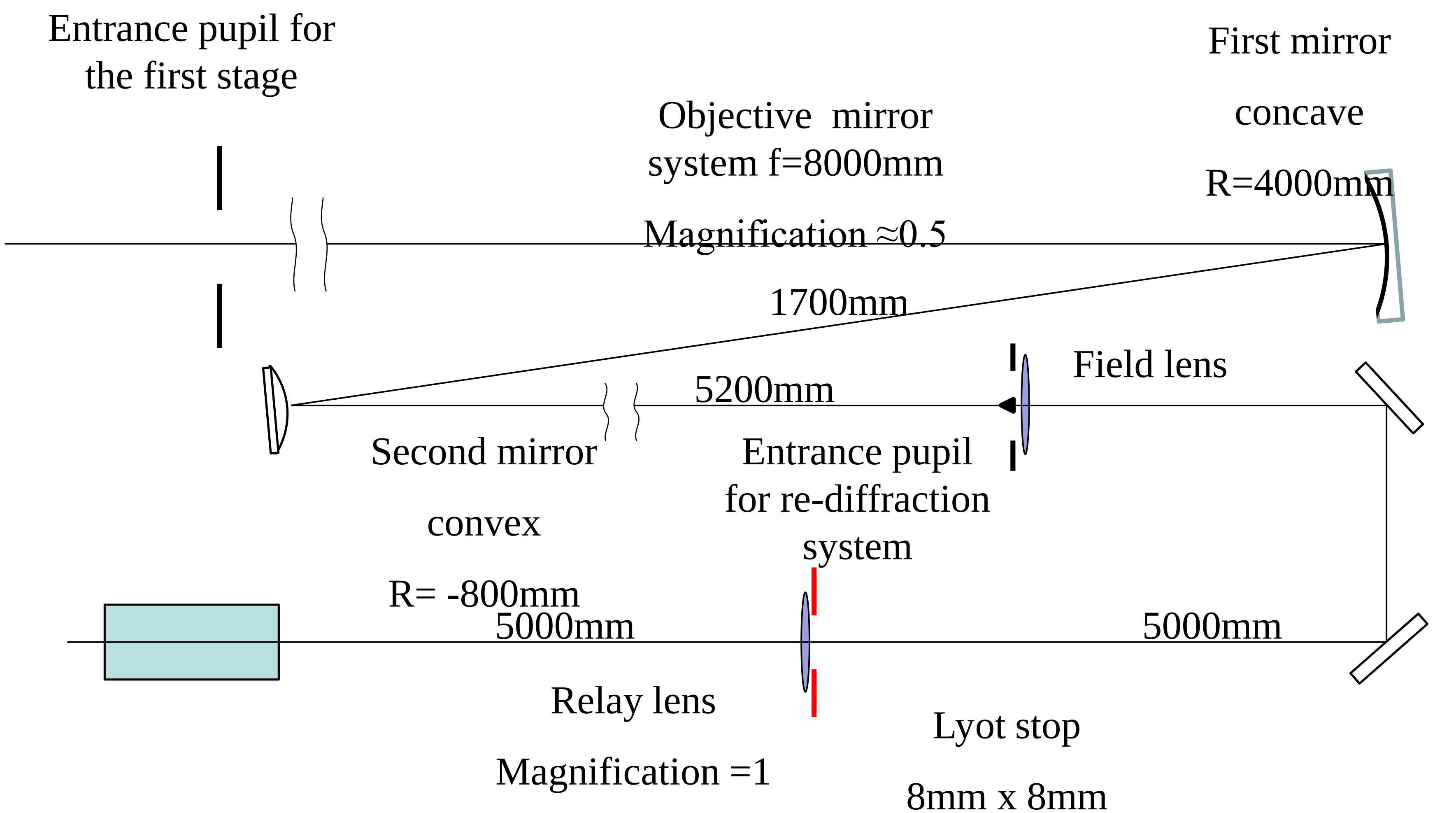
Telephoto type lens



Telephoto type with reflectors

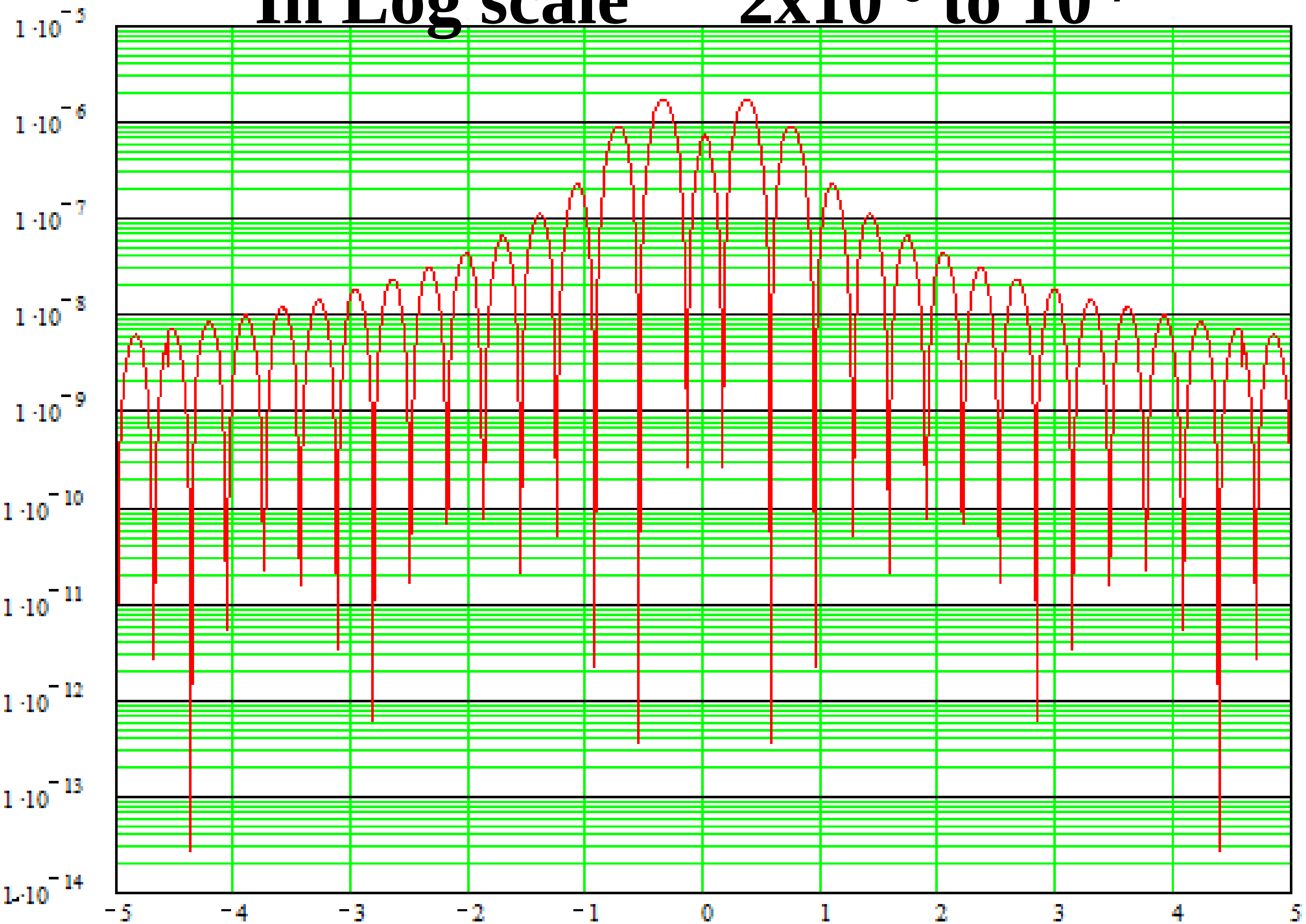


Optical design

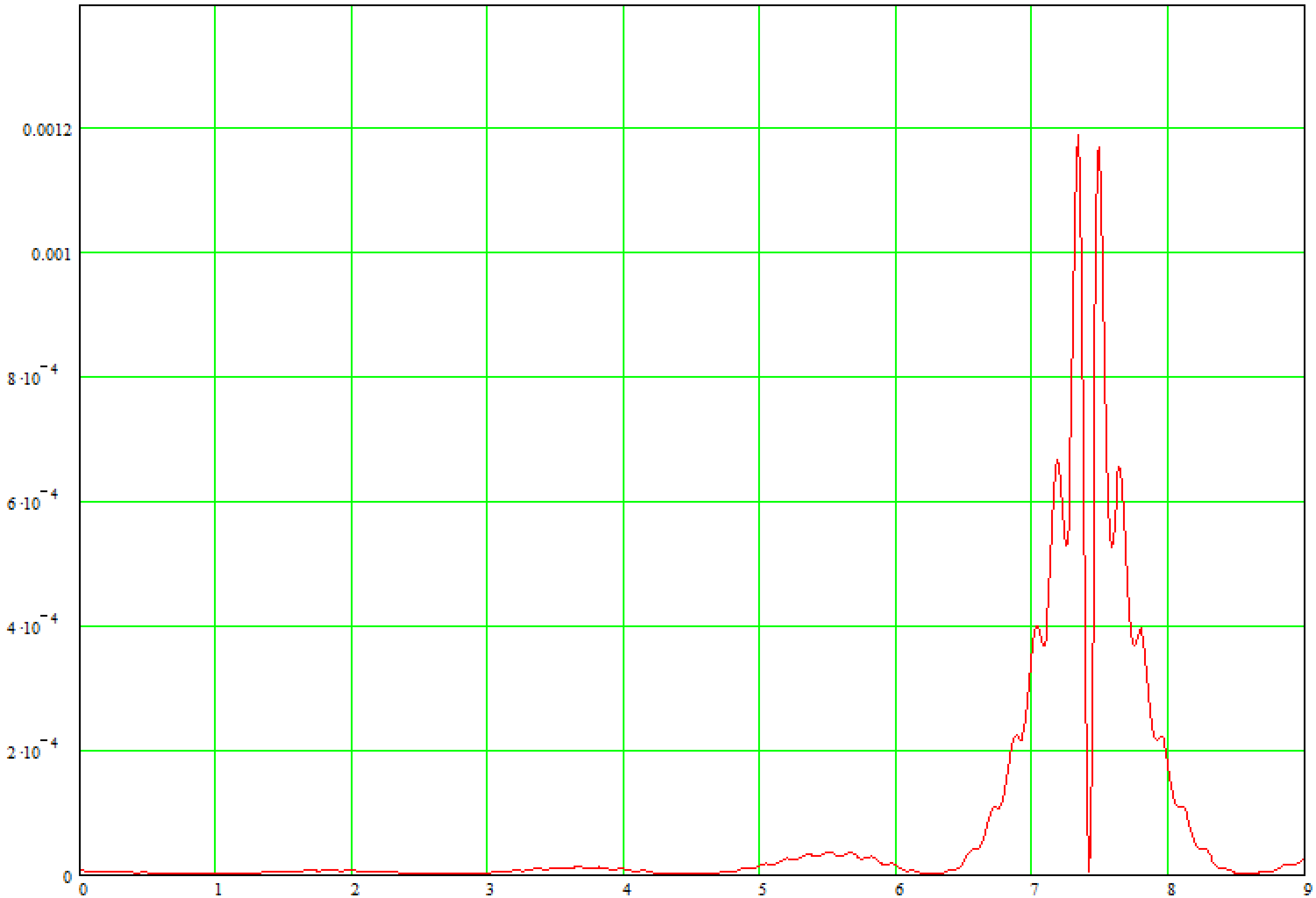


Diffraction background at 3ed stage

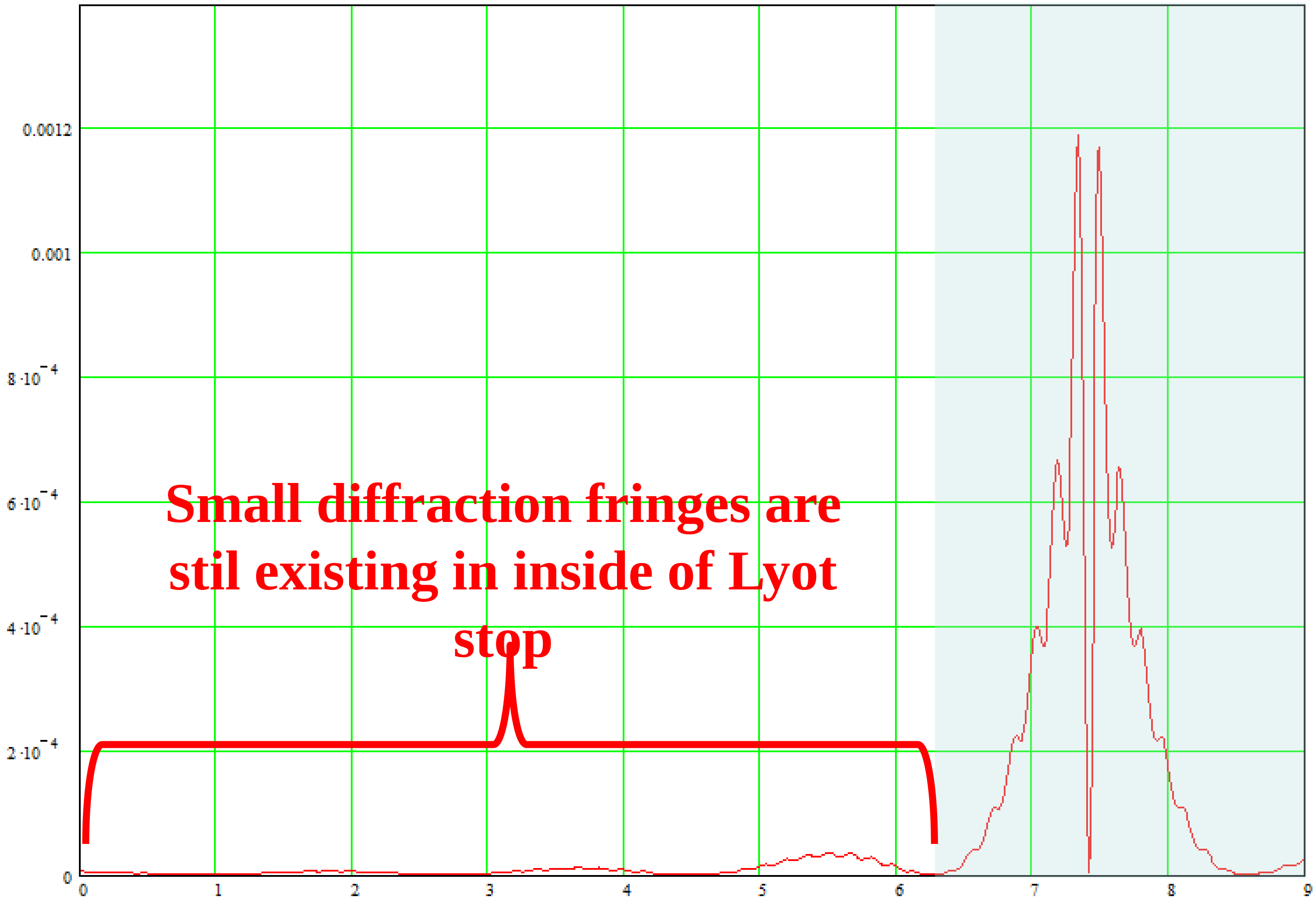
In Log scale $2x10^{-6}$ to 10^{-7}



9. Possibility of appodization to reduce leakage diffraction fringes


0.0012
0.001
8·10⁻⁴
6·10⁻⁴
4·10⁻⁴
2·10⁻⁴
0
0
1
2
3
4
5
6
7
8
9

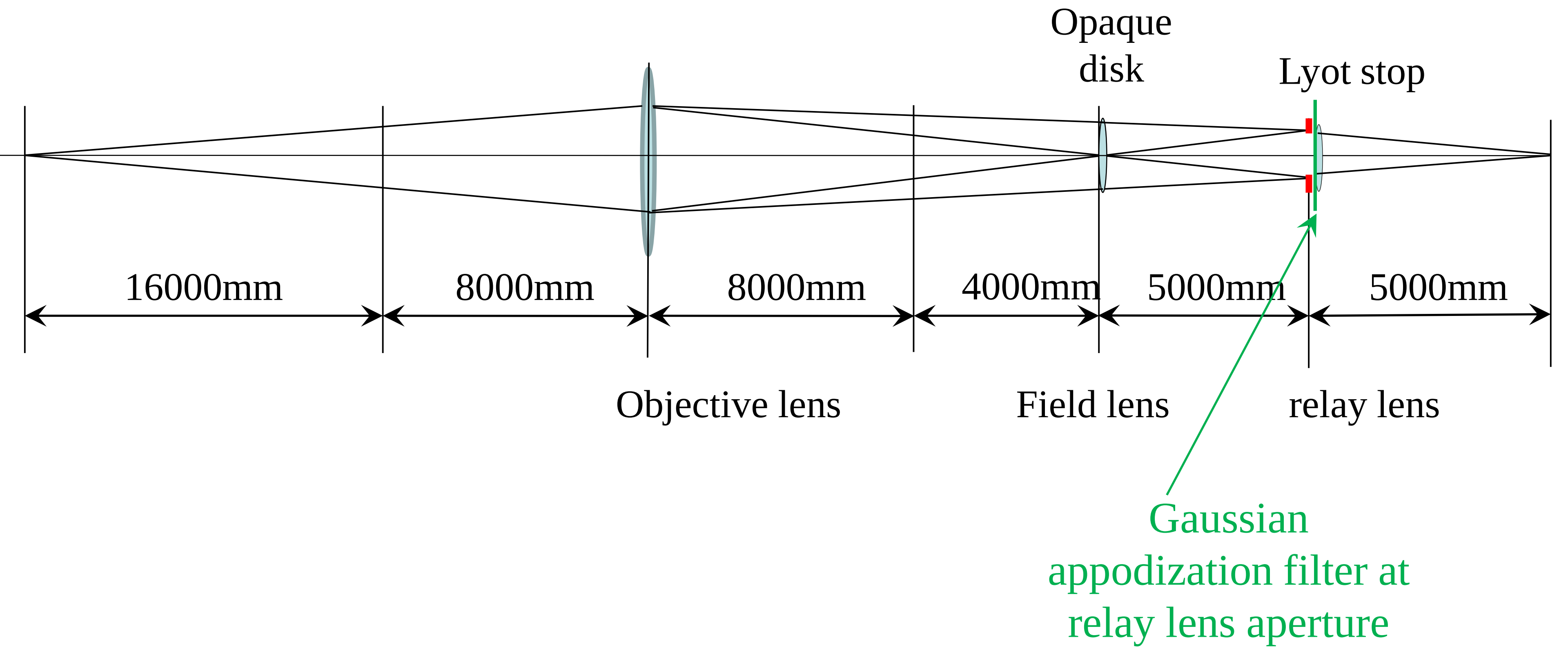
Main diffraction fringe can eliminate by Lyot stop



Appodization at aperture of relay lens .

Since usual appodization is applied to the objective lens aperture.

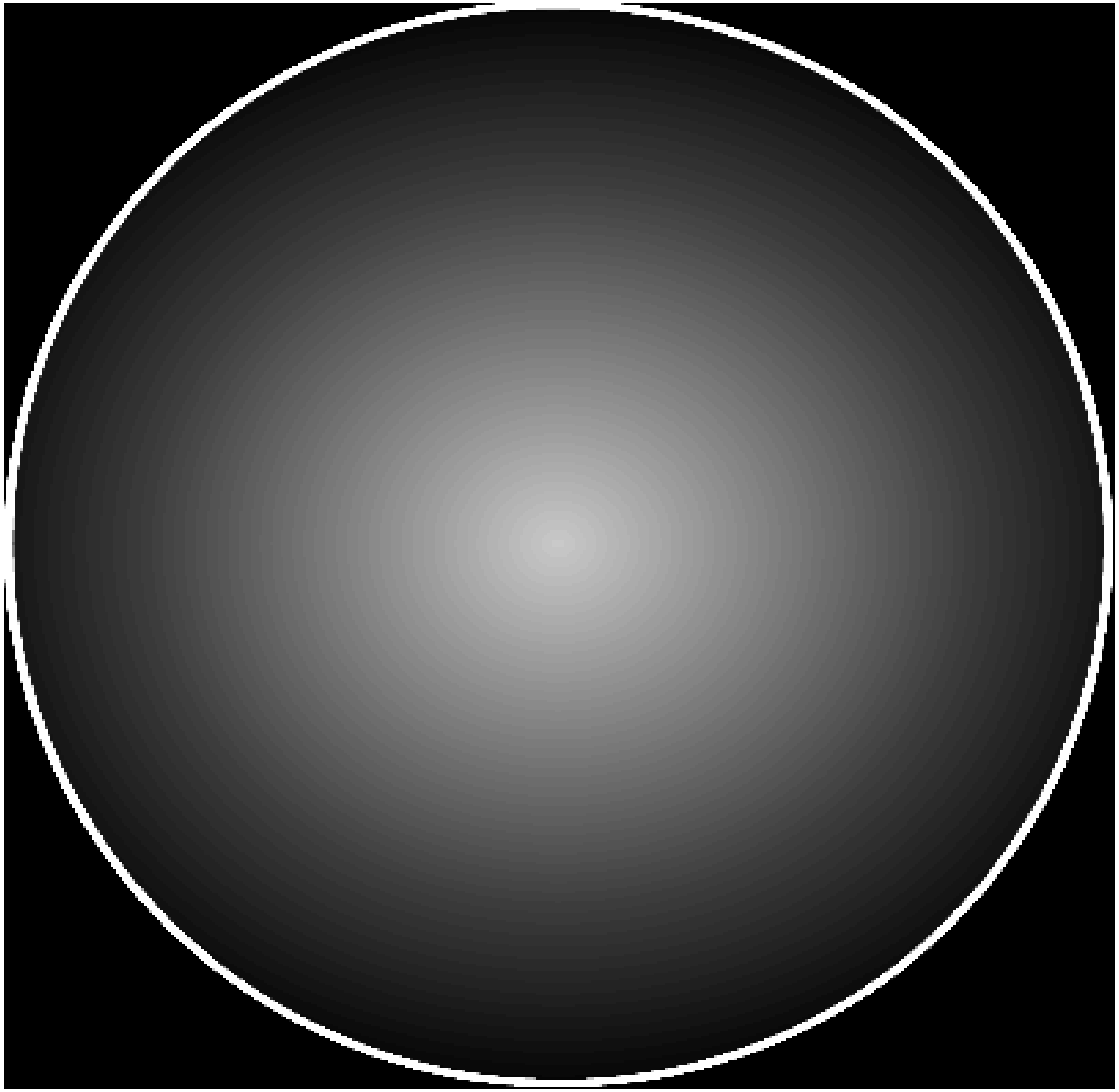
It is not applicable for coronagraph because it's Large Mie scattering noise.



Opaque disk
Lyot stop
16000mm
8000mm
8000mm
4000mm
5000mm
5000mm
Objective lens
Field lens
relay lens
Gaussian appodization filter at relay lens aperture

Appodization of Gaussian graduation

On relay lens aperture



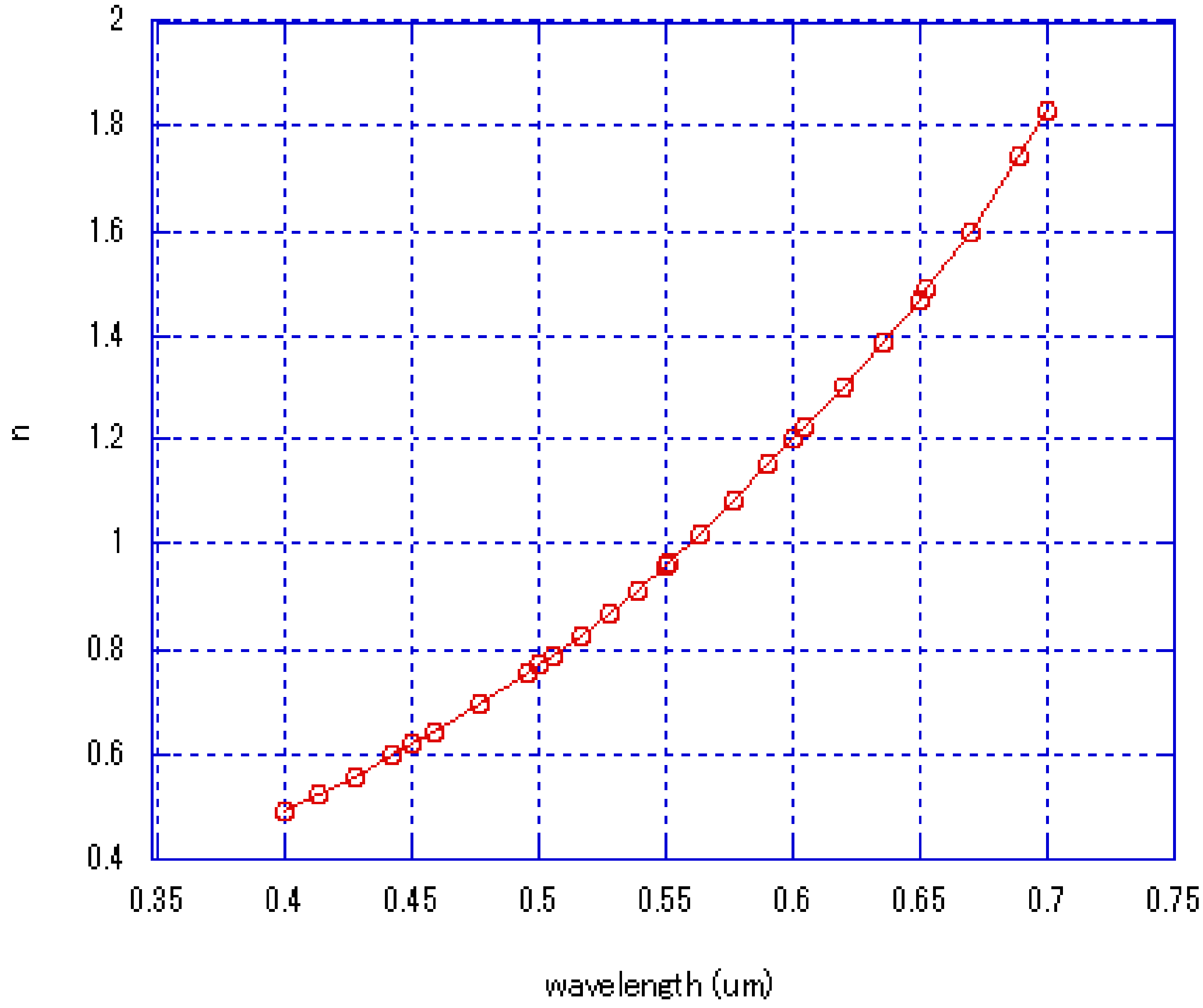
Disturbance of light on final focus point V(x) is given by;

$$\boldsymbol{V}(\phi)=\frac{1}{\boldsymbol{i}\cdot\lambda\cdot f_{relay}}\int_0^{\phi 1}\boldsymbol{u}(\boldsymbol{r})\boldsymbol{W}(\boldsymbol{r})\,\mathbf{exp}\left\{-\frac{\boldsymbol{i}\cdot 2\cdot\pi\cdot\phi\cdot\boldsymbol{r}}{\lambda\cdot f_{relay}}\right\}\boldsymbol{dr}$$

In here, *W*(r) is generalized pupil function of Appodization filter given by,

$$\boldsymbol{W}(r)=\sqrt{I(r)}\,\mathbf{exp}\{-2\pi\,\boldsymbol{i}\,\boldsymbol{k}\,\boldsymbol{n}\cdot l(r)\}$$

Real part of refractive index of Aluminum



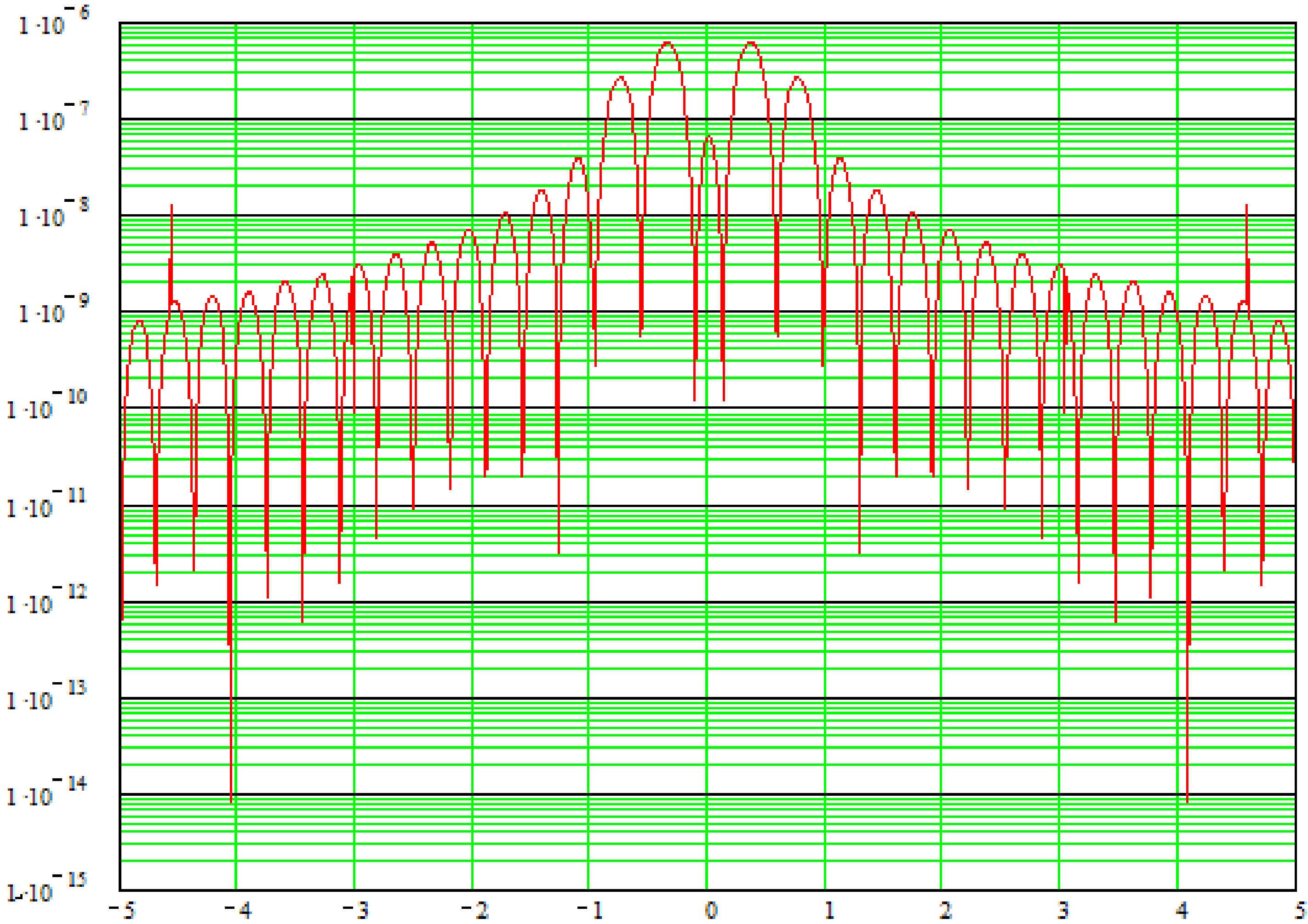
Refractive index is almost 1 at the wavelength of 560nm. So, using this wavelength, appodization filter serves as an amplitude modulation filter, but at other wavelength, it also serves as phase modulation filter.

In shorter and longer wavelength region,

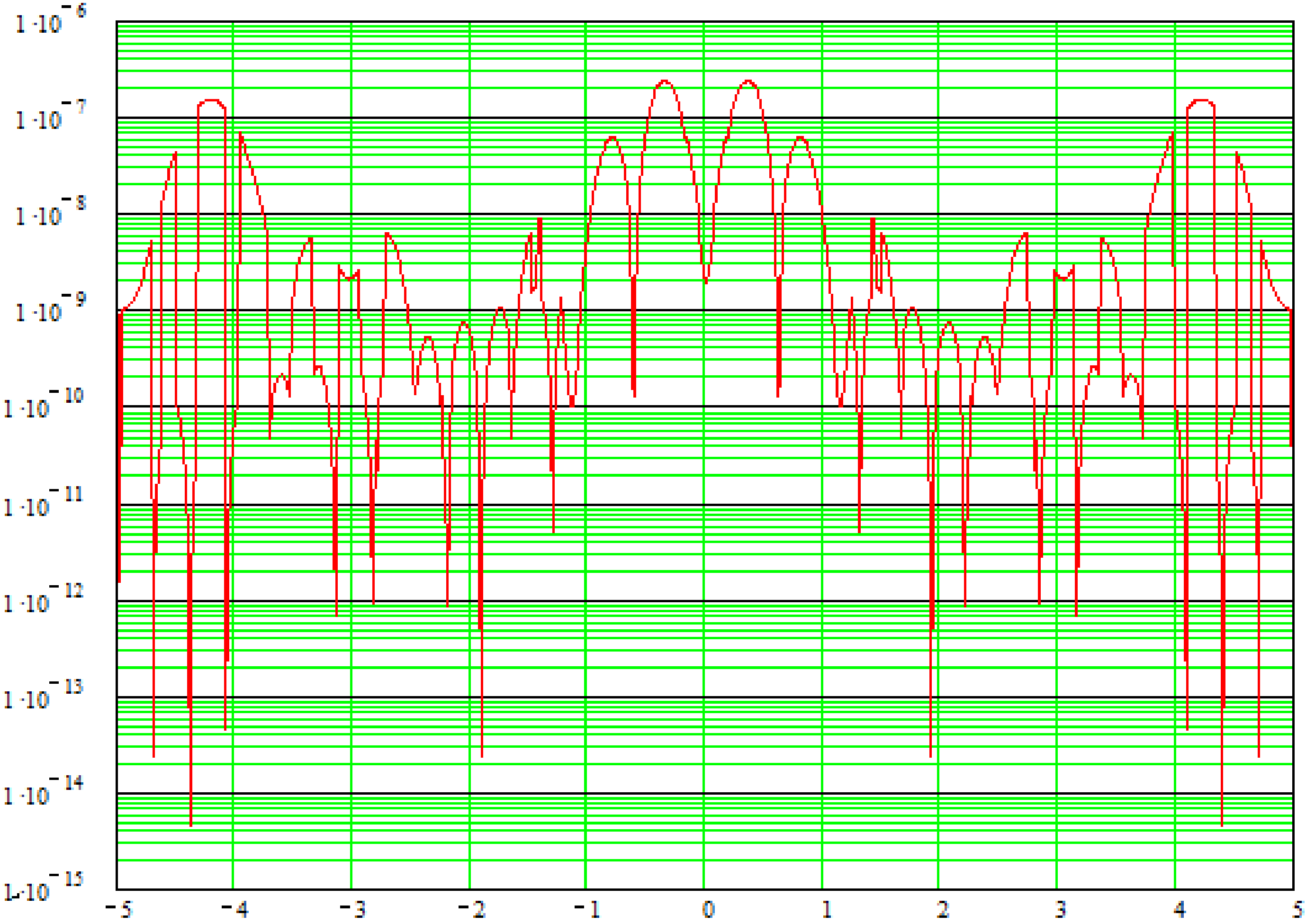
This filter will introduce not negligible phase shift, like spherical aberration.

Some correction for this phase modulation will

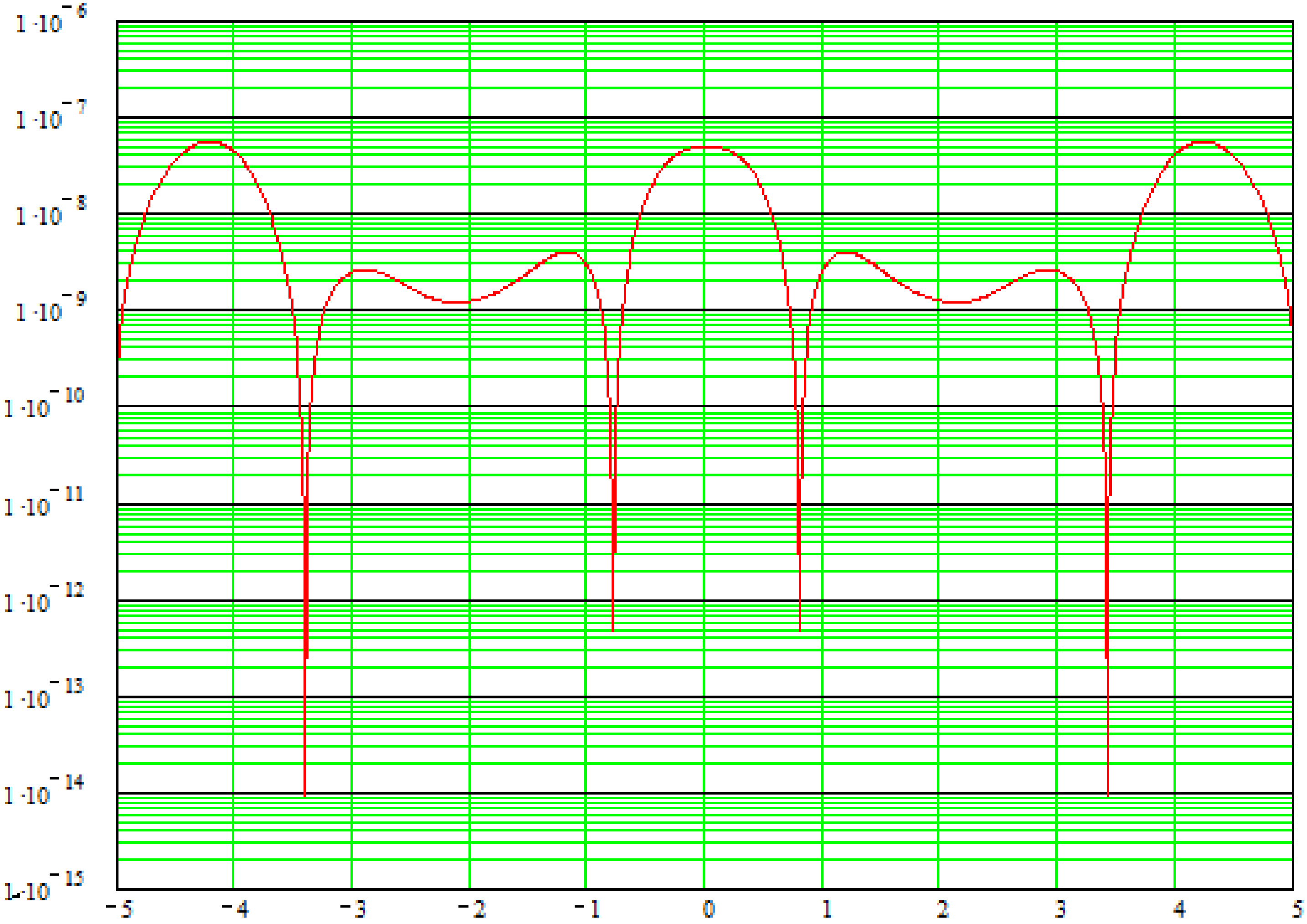
Gaussian appodization a=3mm



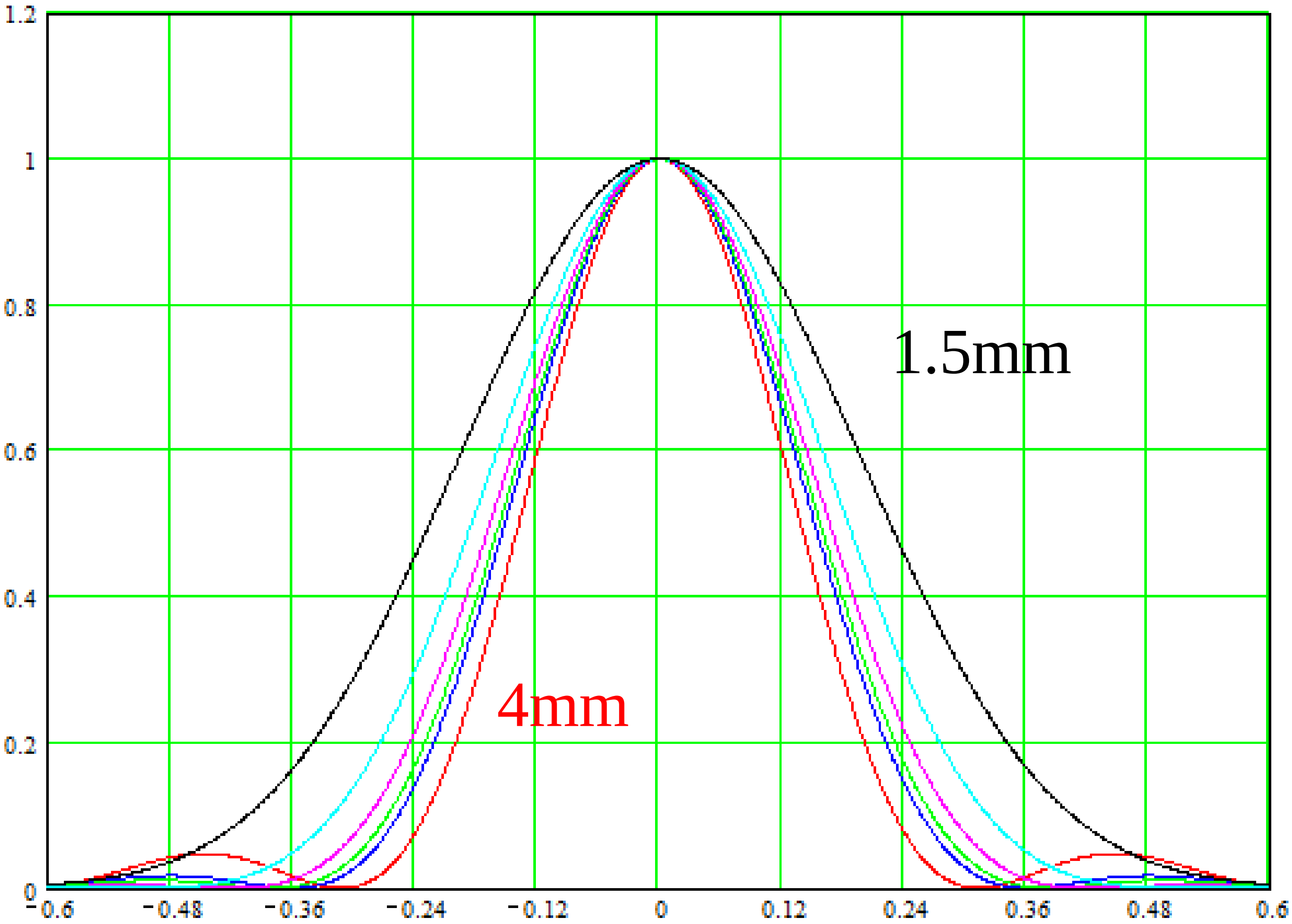
Gaussian appodization a=2mm



Gaussian appodization a=1mm



Reduction of spatial resolution at final image



Expected performance of Phase 2 coronagraph

1. The diffraction background in phase1 coronagraph is estimated to $2x10^{-6}$ to 10^{-7} .

2. Gaussian appodization at relay lens aperture can improve the contrast of noise diffraction down to 10^{-9}.

2. Such appodization increase width of diffraction for halo image, and will reduce spatial resolution.

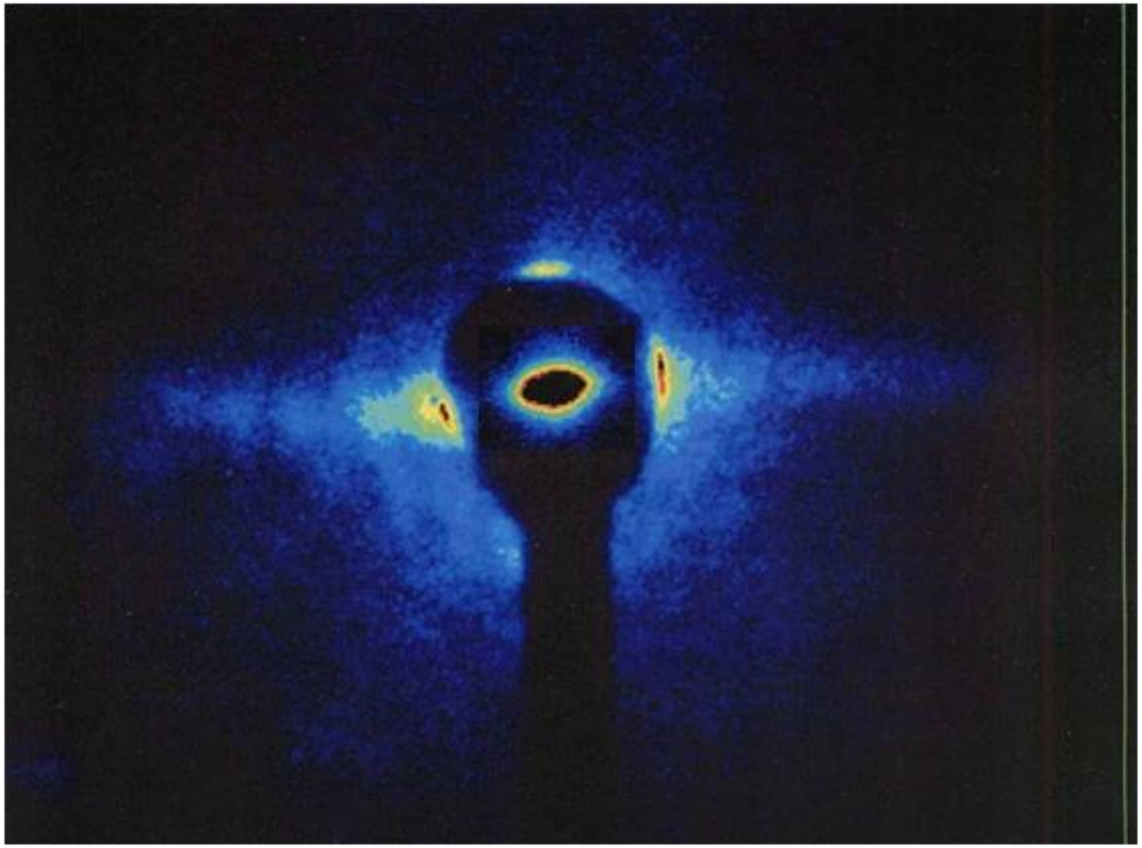
Please note this discussion is only for the theoretical background produced by diffraction phenomena. Reducing practical backgrounds such as Mie scattering from optical components are other issues.

Additional idea for simultaneous observation of beam core and beam halo

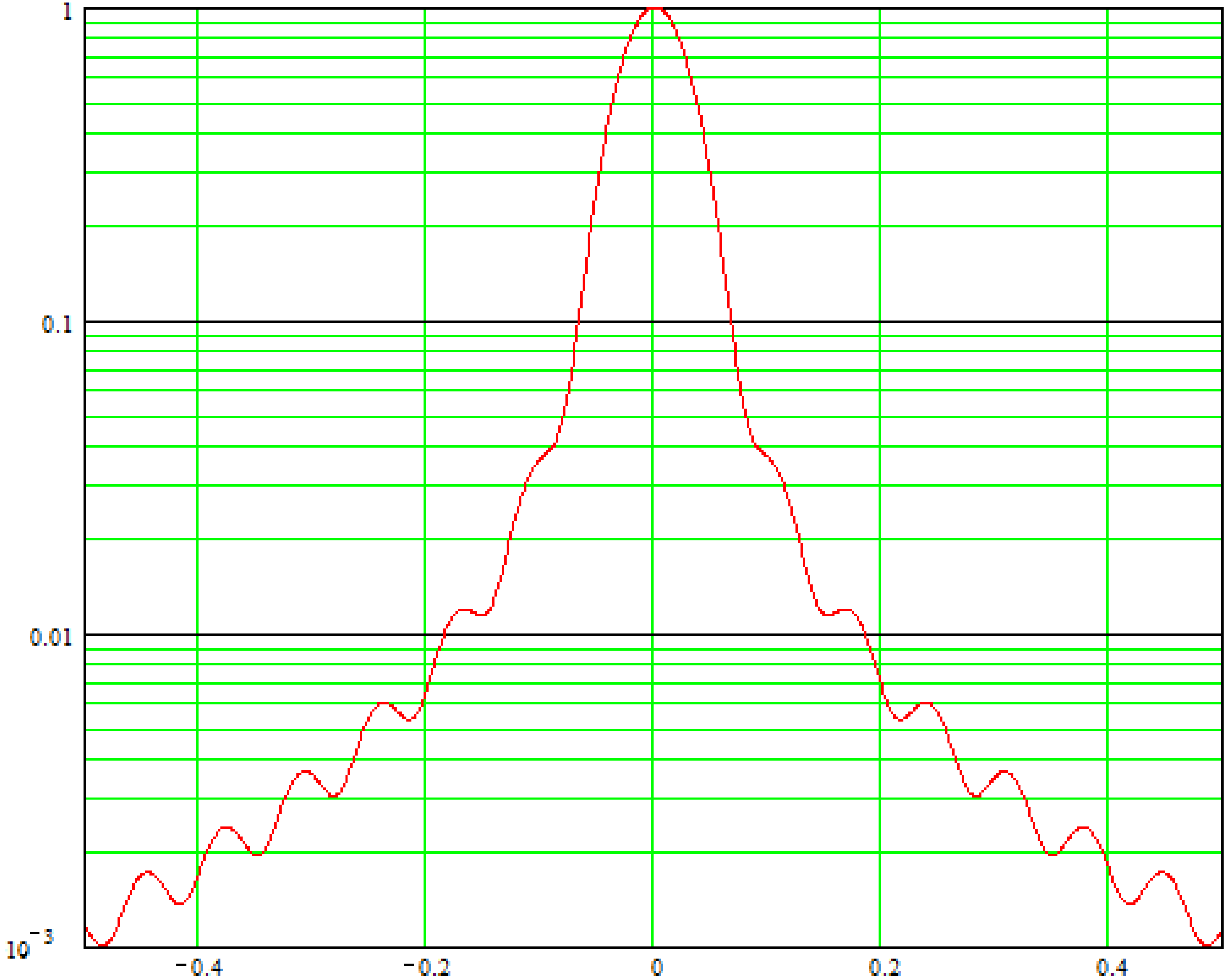
When using 560nm for the observation, Aluminum mask for artificial eclipse give no phase shift.

So, we can apply certain neutral density for this mask, we can also observe beam core image together with beam halo image.

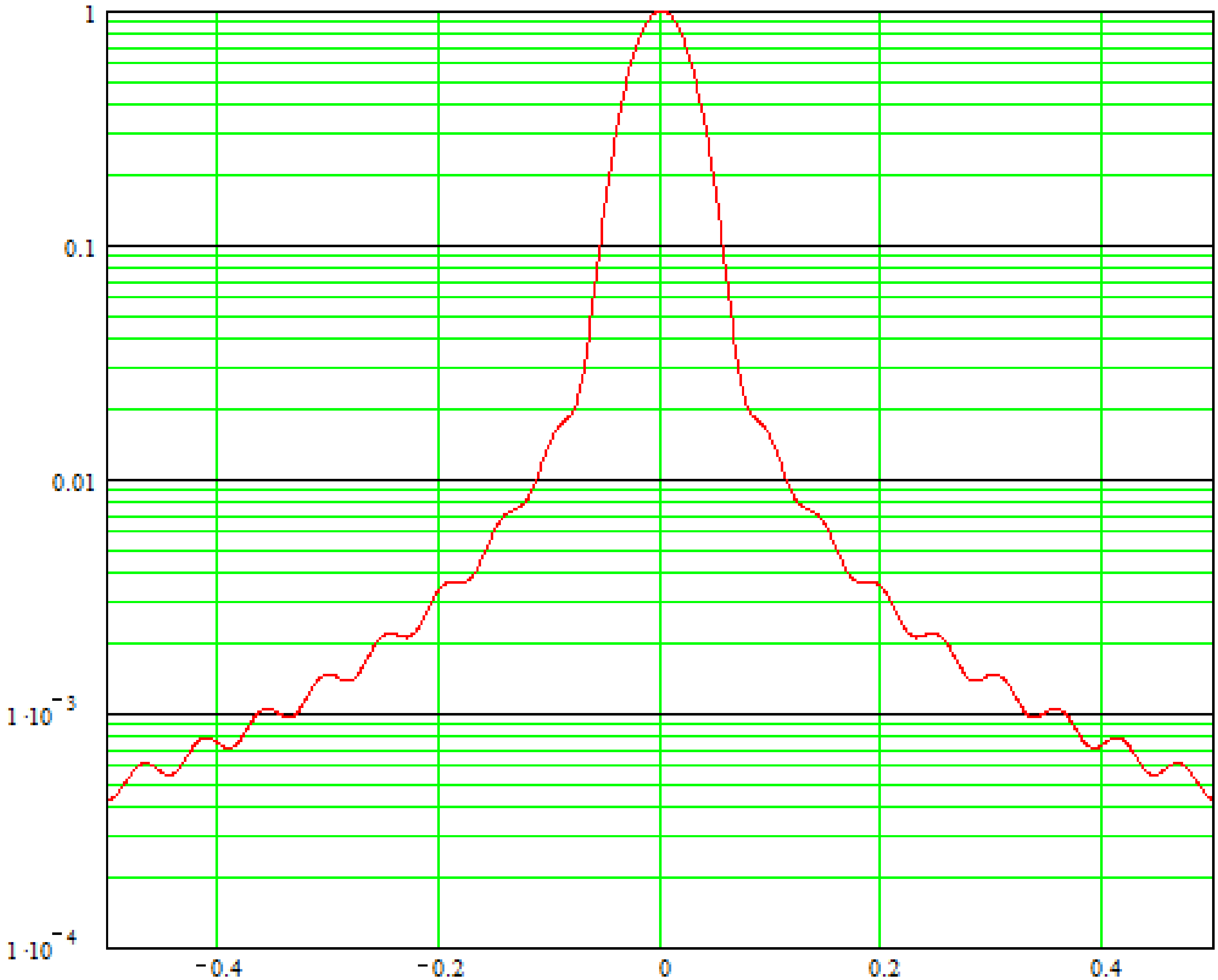
Thank very much for your attention!



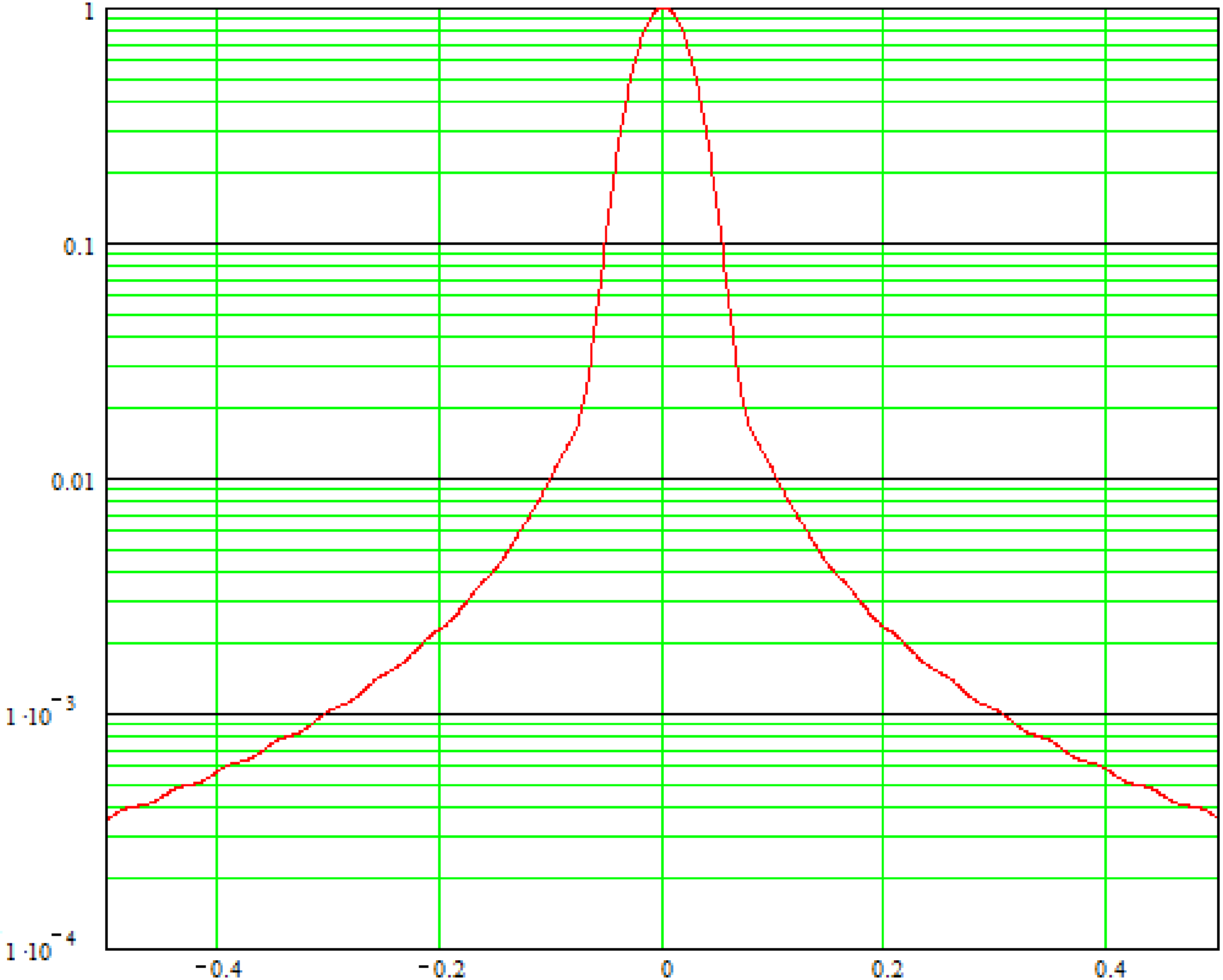
16mm



20mm



24mm



25mm