Accelerator Physics Lecture Modules

1. Introduction/Timing
2. Beam propagation basics
3. Simple Harmonic Oscillator

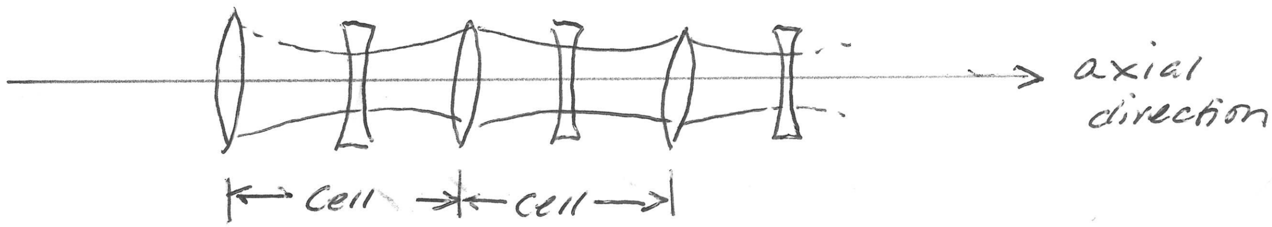
4. Betatron oscillations (transverse motion)
5. Dispersion
6. Closed Orbit Distortion

7. Synchrotron oscillations
8. Properties of synchrotron radiation

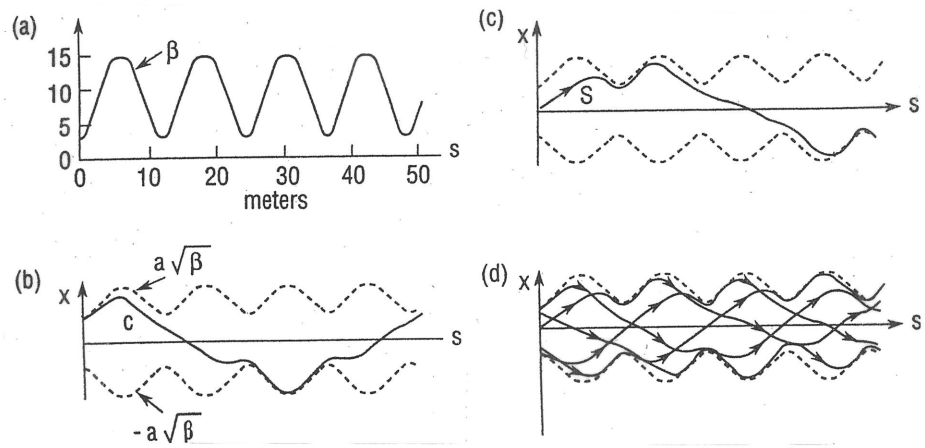
9. Radiation damping
10. Equilibirum emittance

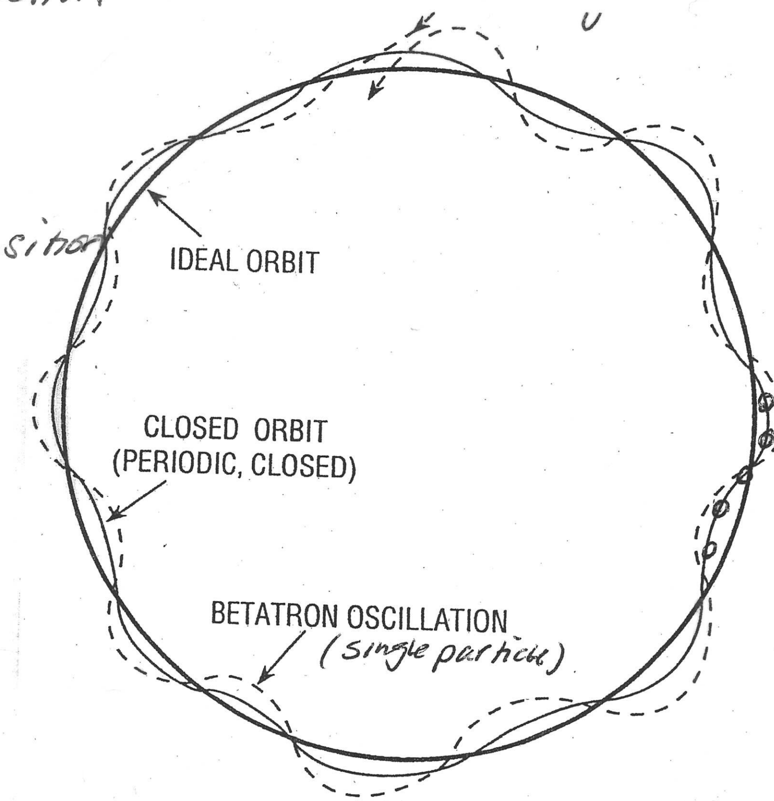
Transverse Single Particle Motion (Betatron Oscillations)

- Betatron oscillations and associated betafunctions describe transverse particle trajectories relative to the design orbit
- Particles are 'kicked' because they collide or emit radiation
- Quadrupoles supply a periodic or quasi-periodic restoring force



- Oscillation amplitude and frequency vary with position





Betatron Oscillations in Phase Space

Previously we had

$$x'' + \overset{\omega^2}{\frac{k}{m}}\, x = 0$$

$$x(t) = A\cos\omega t \triangleq x_0 \cos\omega t$$

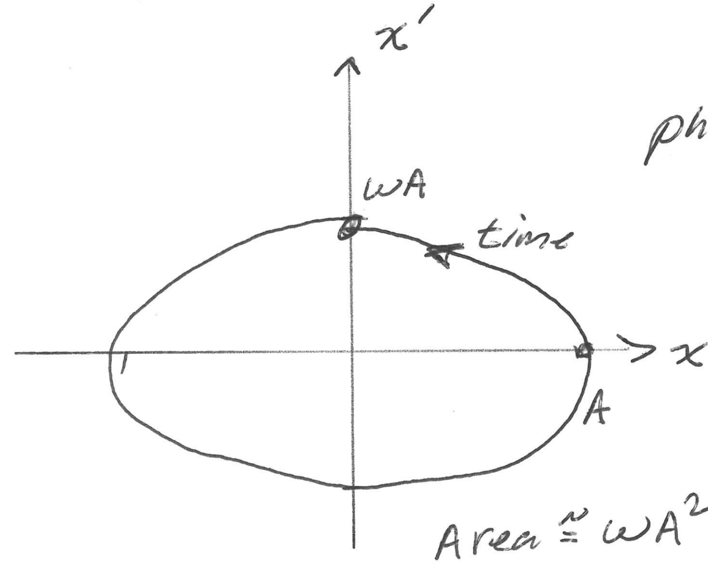
$$x'(t) = -A\omega \sin\omega t \triangleq x_0' \sin\omega t$$

A = Amplitude = const

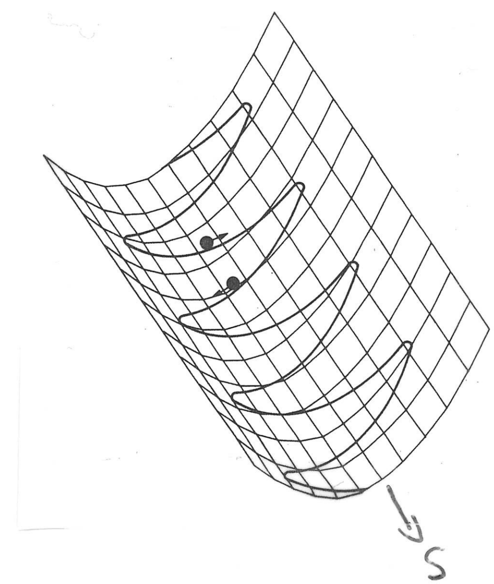
ϕ = phase = $\int \omega\, dt$ ω = const

$$x_0 x_0' = \omega A^2 = \epsilon$$

- action
- energy of oscillation
- area of phase ellipse



phase space



Ball in <u>constant</u> gutter

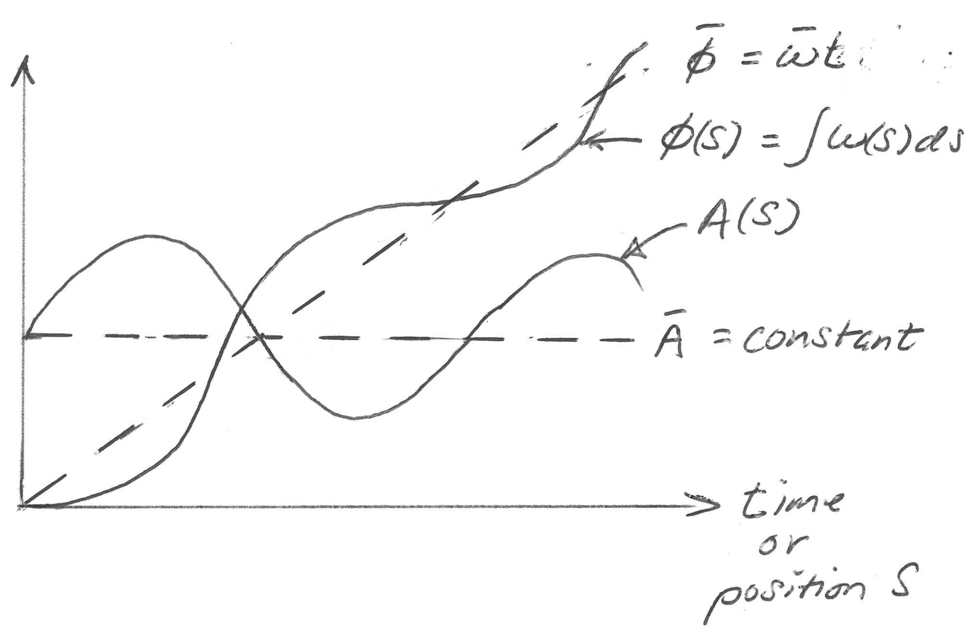
Betatron Oscillations

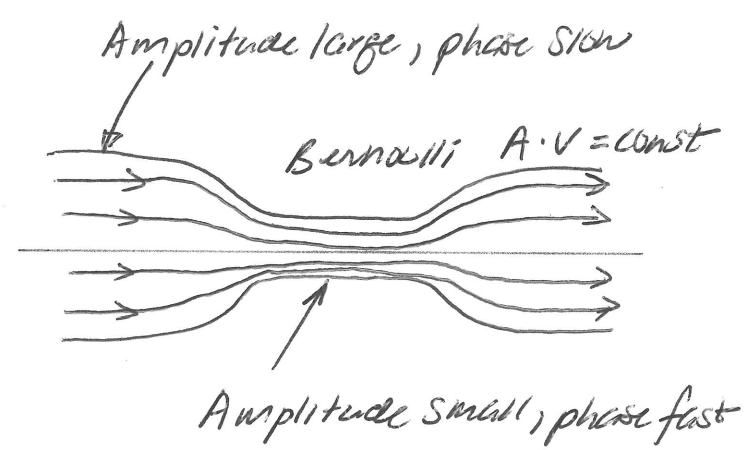
For accelerators the restoring force varies in position

$$x'' = \frac{d^2x}{ds^2} + \sqrt{\frac{K(s)}{m}}\, x = 0$$

axial direction

Particle amplitude and phase vary axially





Betatron Oscillations – The "Solution"

$$\frac{d^2x}{ds^2} + K^2(s)\,x = 0$$

<u>try</u> $x(s) = \sqrt{\epsilon}\cdot\sqrt{\beta(s)}\cdot\cos(\phi(s))$

$\sqrt{\epsilon}$ = amplitude scale factor

$\sqrt{\beta(s)}$ = amplitude modulation

$$\frac{dx}{ds} = \sqrt{\frac{\epsilon}{\beta}}\left[\frac{\beta'\cos\phi}{2} + \sin\phi\right] \quad \text{(two terms)}$$

$$\frac{d^2x}{ds^2} = \text{"something with } \cos\phi \text{ and } \sin\phi \text{ terms"}$$

Substitute
Collect terms

$$\frac{d^2x}{ds^2} + K^2(s)\,x = \left[(\beta,\phi)\right]\sin\phi + \left[(\beta,\phi)\right]\cos\phi$$

Coefficients independently $= 0$

$$\phi'\beta = \text{const}$$

$$\boxed{\phi = \int \frac{ds}{\beta}}$$

phase depends on β

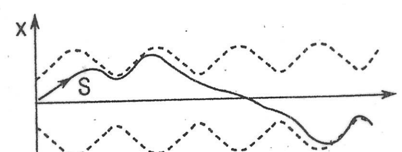
$$\boxed{\frac{\beta''}{2} - \frac{(\beta')^2}{4\beta} - \frac{1}{\beta} + K^2\beta = 0} \quad (!)$$

Non-linear D.Eq.

Note: if $\beta' = 0$

then $\frac{1}{\beta} = K^2\beta$

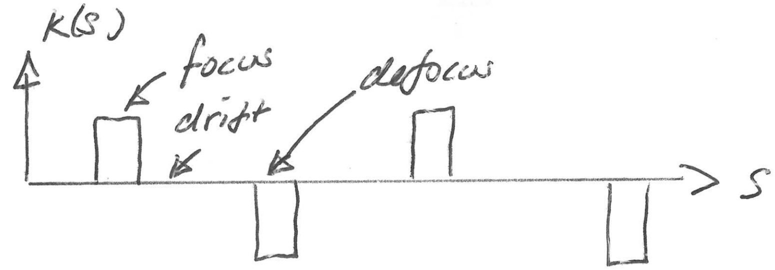
$K^2 = \frac{1}{\beta^2}$ Constant restoring force



Betatron Oscillations – Envelope

We have a valid solution

$$\frac{d^2x}{ds^2} + K^2(s)\,x = 0$$



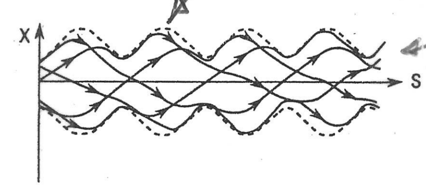
$$x(s) = \underbrace{\sqrt{\epsilon}\,\sqrt{\beta(s)}}_{\text{amplitude}}\ \cos \underbrace{\phi(s)}_{\text{phase}}$$

piecewise continuous function K(s)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \cdots M_4 M_3 M_2 M_1 \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$\sqrt{\epsilon\beta(s)}$ is the <u>envelope</u> <u>function</u>

$\sqrt{\epsilon\beta}$ = beam envelop



particles with different <u>phases</u> make up "the beam"

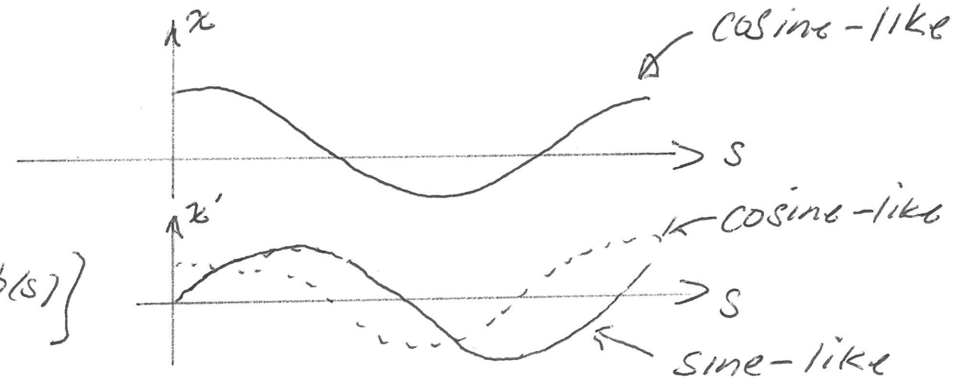
$$f(x,x') = C \cdot e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{xx'}{2\sigma_x\sigma_{x'}} + \frac{x'^2}{2\sigma_{x'}^2}\right)}$$

σ_x and $\sigma_{x'}$ vary with s-position

$$\phi = \int \frac{ds}{\beta} + \Phi_0 \text{ for all particles}$$

Betetron Oscillations – Phase Space Ellipse

$$\frac{d^2x}{ds^2} + k^2 x = 0$$



$$x(s) = \sqrt{\epsilon\beta(s)} \cdot \cos(\phi(s))$$

$$x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left[\frac{\beta'}{2}\cos\phi(s) + \sin\phi(s)\right]$$

As before eliminate phase ϕ to derive ellipse equation

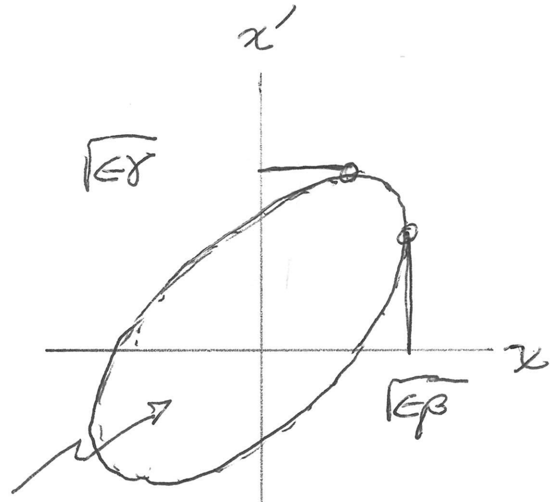
$$\left[\frac{1}{2}\left(1+\left(\frac{\beta'}{2}\right)^2\right)\right]x^2 + \left[2\frac{\beta'}{2}\right]xx' + \beta x'^2 = \epsilon$$

Conventionally $\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon}$

(α, β, γ) collectively "Twiss parameters"

ϵ = emittance = "Courant–Snyder Invariant"

$$\alpha \triangleq \frac{\beta'}{2} \qquad \gamma \triangleq \frac{1+\alpha^2}{2}$$



ellipse area $= \epsilon$

$x_0 \cdot x_0' = \sqrt{\epsilon\beta} \cdot \sqrt{\frac{\epsilon}{\beta}} = \epsilon$

Single particle oscillation amplitude

Betatron Oscillations – Phase Space Transport

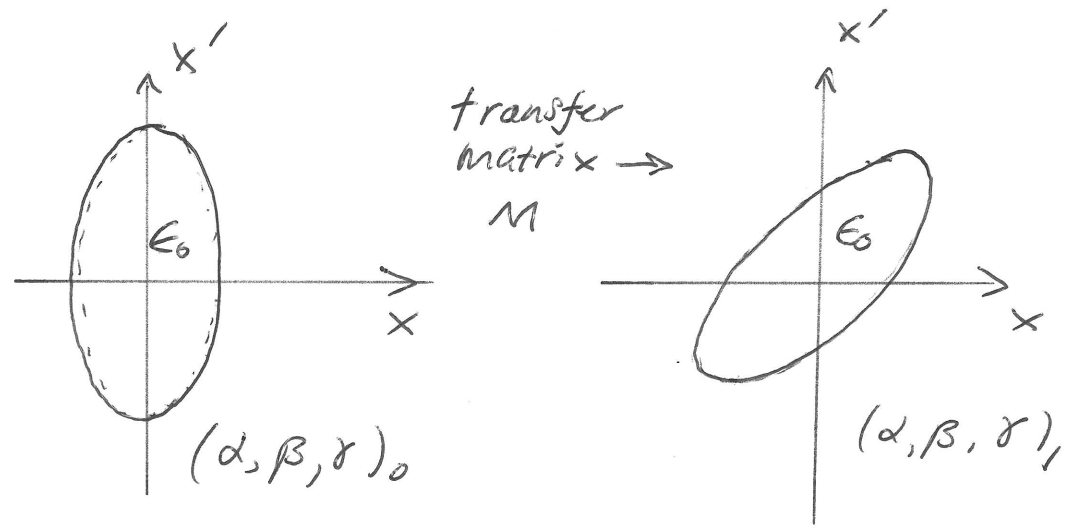
$$\frac{d^2x}{ds^2} + k^2(s)\,x = 0$$

$$\begin{cases} x(s) = \sqrt{\epsilon\beta(s)}\cos\phi(s) \\ x'(s) = \cdots \end{cases}$$

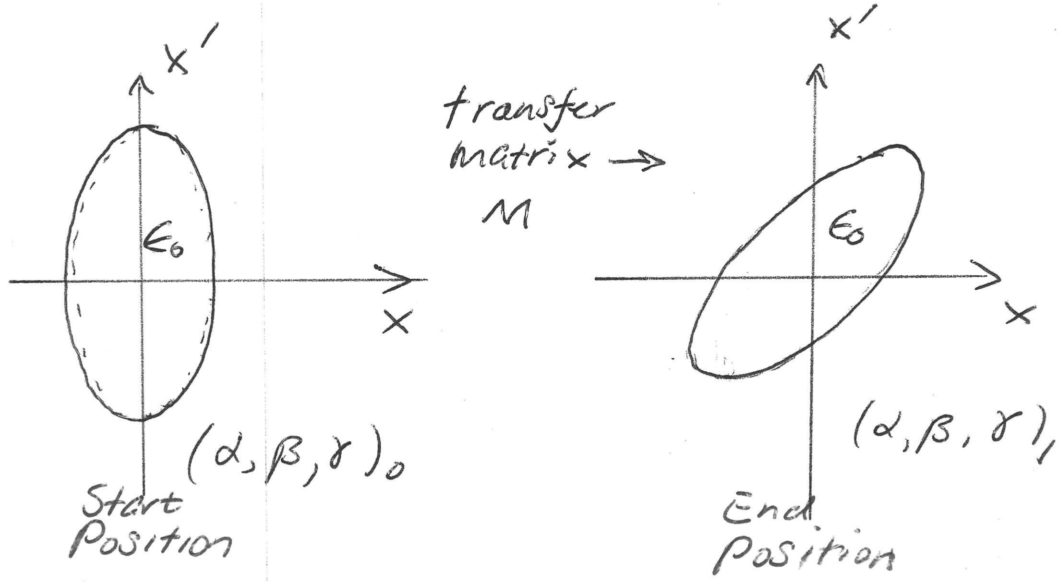
Ellipse $\quad \gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$

where $\frac{\beta''}{2} - \frac{(\beta')^2}{4\beta} - \frac{1}{\beta} + k(s)\beta = 0 \quad , \quad \alpha = -\frac{\beta'}{2} \quad , \quad \gamma = \frac{1+\alpha^2}{\beta}$

$\Rightarrow \beta$ and therefore (α, β, γ) evolve with s (depending on $k(s)$)



Beam Ellipse Transport



$$\gamma_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \beta x_0^2 = \epsilon_0 = \gamma_1 x_1'^2 + 2\alpha x_1 x_1' + \beta x_1^2$$

where $\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M \begin{pmatrix} x \\ x' \end{pmatrix}_0 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 \Rightarrow \begin{aligned} x_1 &= m_{11} x_0 + m_{12} x_0' \\ x_1' &= m_{21} x_0 + m_{22} x_0' \end{aligned}$

Substite, terms mix, collect

$$\boxed{\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_1 = \begin{pmatrix} \text{3×3} \\ \text{ellipse} \\ \text{rotation} \\ \text{matrix} \\ R \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_0}$$

$$R_{total} = \cdots R_3 \cdot R_2 \cdot R_1$$

Beam Ellipse Transformation

A more sophisticated approach

$$\gamma_1 x_1'^2 + 2\alpha_1 x_1 x_1' + \beta_1 x_1'^2 = \epsilon = (x \;\; x')_1 \begin{pmatrix} \beta & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1 = \vec{x}_1^T \Sigma_1 \vec{x}_1$$

Beam Sigma Matrix

Similar to before $\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$

$$\vec{x}_0^T \Sigma_0 \vec{x}_0 = \epsilon = (M^{-1}\vec{x}_1)^T \cdot \Sigma_0 \cdot (M^{-1}\vec{x}_1)$$

$$= \vec{x}_1^T \Sigma_1 \vec{x}_1$$

$$\boxed{\Sigma_1 = M \Sigma_0 M^T}$$

Similarity transformation preserves invariant ϵ

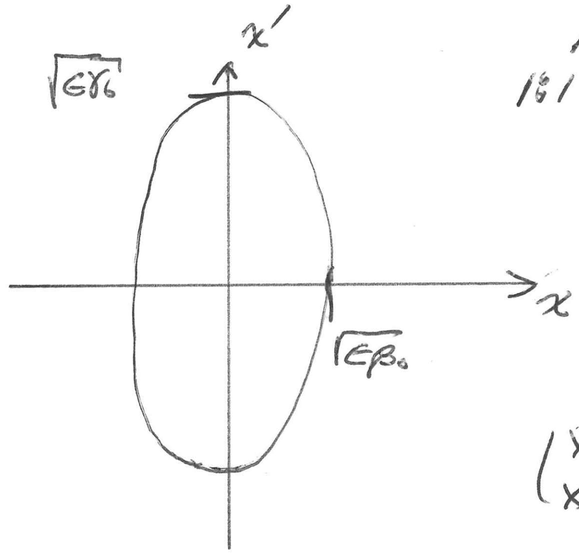
$$\Sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = cov^{-1}$$

$$cov = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

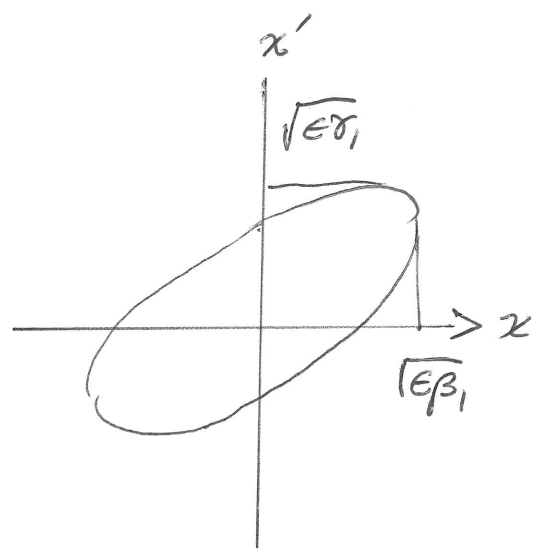
Bi-Gaussian Distribution Function

Bi-Gaussian $\quad f(x, x') = A\, e^{-\frac{(\gamma x^2 + 2\alpha x x' + \beta x'^2)}{\epsilon_0}}$

$f(x, x') = \text{constant} \rightarrow$ defines an ellipse
(numerator constant)



M
1:1 linear map

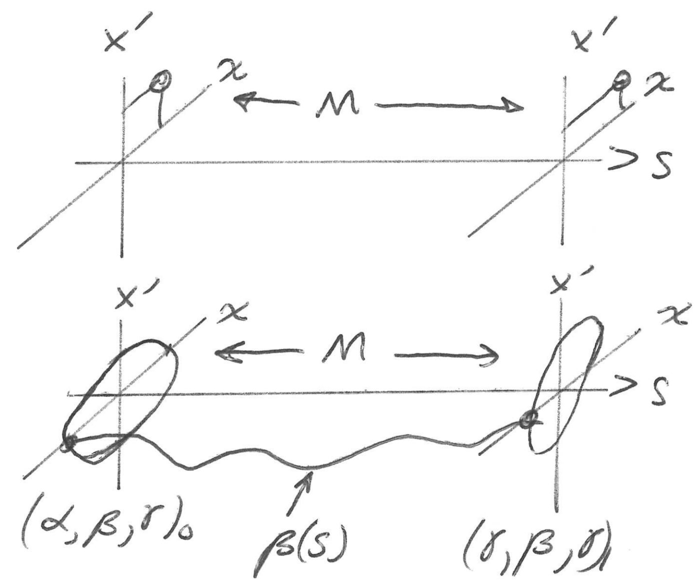


$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$\Sigma_0 = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \qquad \Sigma_1 = M \Sigma_0 M^T \qquad \Sigma_1 = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_1$$

$$\phi = \int \frac{ds}{\beta}$$

Single Particle Transport in terms of Amplitude and Phase



For a single particle

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Equivalently

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\phi(s)$$

$$x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\frac{\beta'}{2}\cos\phi(s) + \sin\phi(s)\right)$$

<u>Start</u>

$$x_0 = \sqrt{\epsilon\beta_0}\cos\phi_0$$

$$x_0' = \sqrt{\frac{\epsilon}{\beta_0}}\left(\frac{\beta_0'}{2}\cos\phi_0 + \sin\phi_0\right)$$

<u>End</u>

$$x_1 = \sqrt{\epsilon\beta_1}\cos(\phi_0 + \Delta\phi)$$

$$x_1' = \sqrt{\frac{\epsilon}{\beta_1}}\left(\frac{\beta_1'}{2}\cos(\phi_0+\Delta\phi) + \sin(\phi_0+\Delta\phi)\right) \leftarrow \text{Expand trig functions}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \underbrace{\begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}\cos\Delta\phi + \frac{\beta_0'}{2}\sin\Delta\phi & \sqrt{\beta_0\beta_1}\sin\Delta\phi \\ \frac{-(1-\alpha_0\alpha_1)\sin\Delta\phi + (\alpha_1-\alpha_0)\cos\Delta\phi}{\sqrt{\beta_0\beta_1}} & \sqrt{\beta_0/\beta_1}\,(\cos\Delta\phi - \alpha_1\sin\Delta\phi) \end{pmatrix}}_{M} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

<u>Why are Betafunctions Important?</u>

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

describes single particle linear optics
- ray tracing

$$x(s) = \sqrt{\epsilon\beta}\cos\phi$$
$$x'(s) = \sqrt{\frac{\epsilon}{\beta}}\left(-\alpha\cos\phi + \sin\phi\right)$$

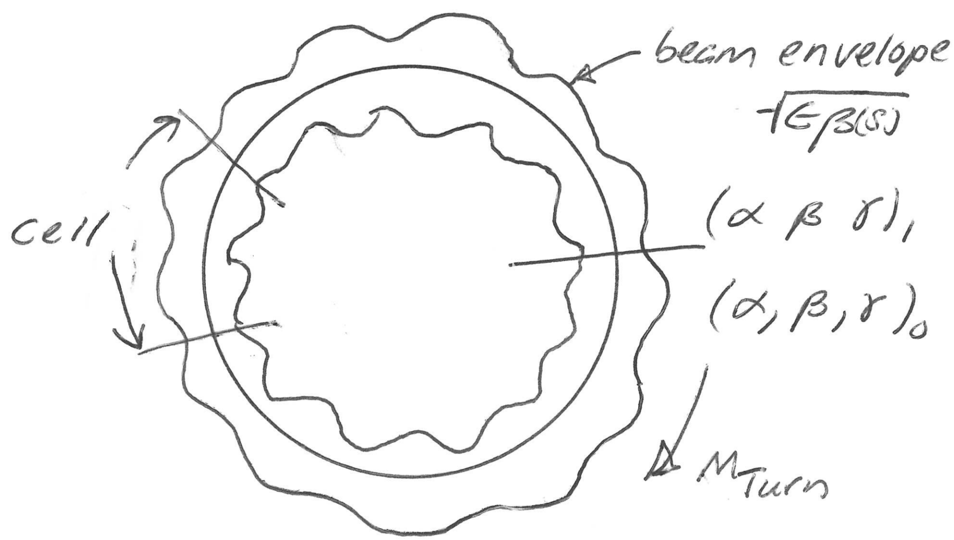
describes single particle optics <u>and</u> the beam envelope

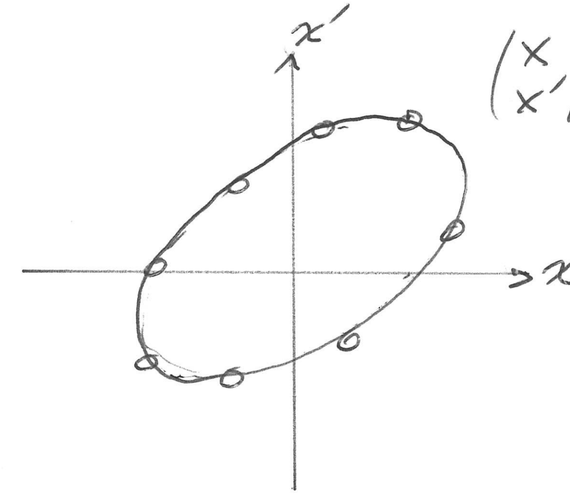
We are interested in both

- single particle dynamics for beam stability, etc.
- beam envelope for physics applications
  - Collider
  - light source
  - Medical
  - welding
  - etc

Periodic Solutions

For a Storage Ring or cell the beam envelope repeats





$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$\Sigma_1 = \Sigma_0 = M_{Turn} \cdot \Sigma_0 \cdot M_{Turn}^T = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \quad \text{or} \quad M_{cell} \Sigma_0 M_{cell}^T = \Sigma_1$$

Solve
$$\begin{cases} \beta_0 = \dfrac{2m_{12}}{\sqrt{2 - m_{11}^2 - m_{12}m_{21} - m_{22}^2}} \longrightarrow \text{need } \sqrt{} > 0 \\ \alpha_0 = \dfrac{m_{11} - m_{22}}{2m_{12}} \beta_0 \\ \gamma_0 = \dfrac{1 + \alpha_0^2}{\beta_0} \end{cases}$$

Examples of Betafunctions

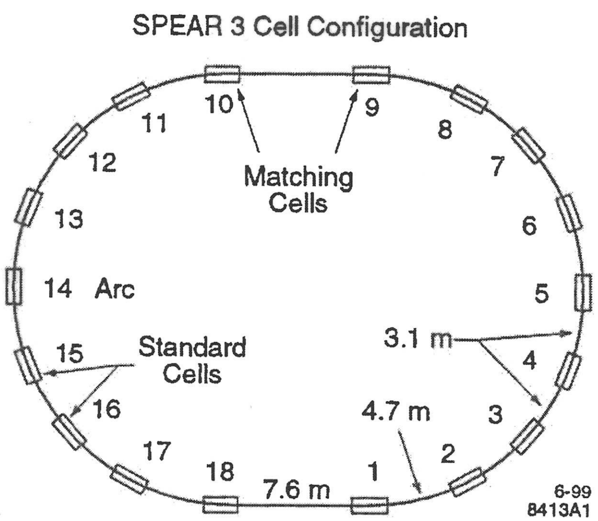
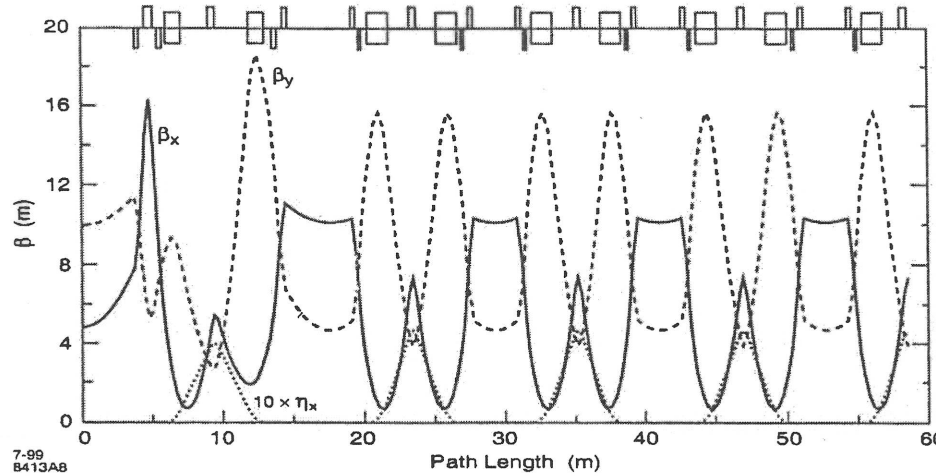
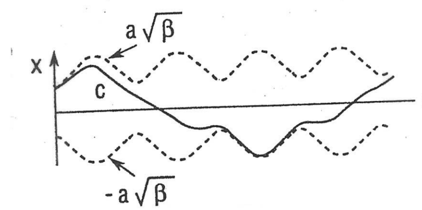


Figure 3.1 SPEAR storage ring configuration.



Note betafunctions follow the magnet lattice $K(s)$
Peaks are not separated by a betatron wavelength



$\bar{\beta}_x \sim 4$

$\epsilon_x \sim 10^{-8}$

$\sqrt{\epsilon_x \beta_x} = 2 \times 10^{-4}\,m$ (200 μm)

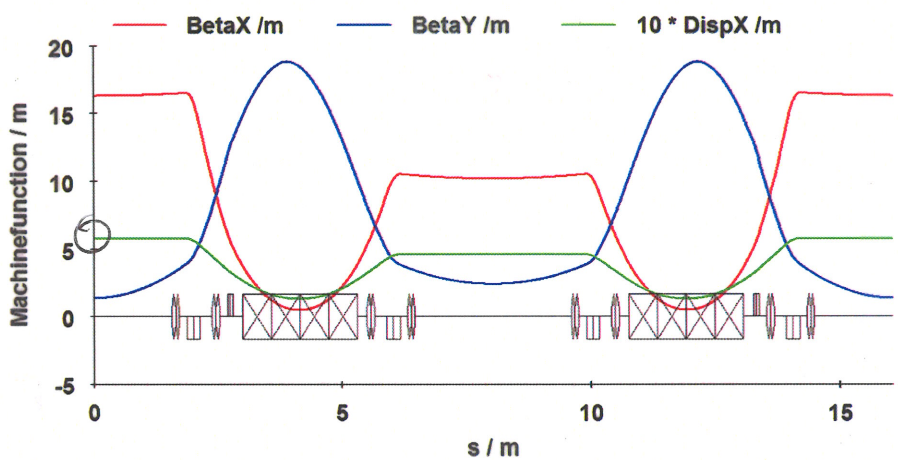
$\bar{\beta}_y \sim 4$

$\epsilon_y \sim 10^{-10}$

$\sqrt{\epsilon_y \beta_y} = 2 \times 10^{-5}\,m$ (20 μm)

Examples of Single Cells

Double-Bend 'Achromat'



$\epsilon_x \sim \theta^3$

Multi-Bend Achromat

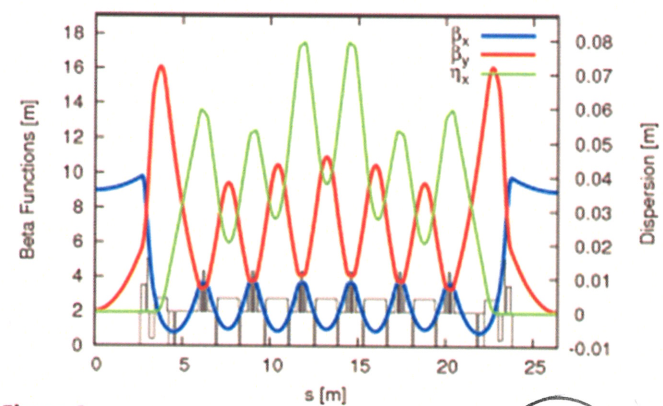


Figure 3
β functions and dispersion for one achromat of the MAX IV 3 GeV storage ring. Magnet positions are indicated at the bottom.

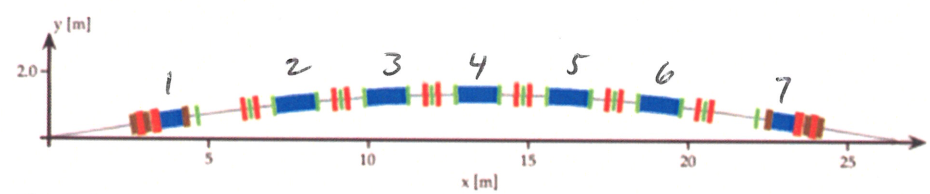


Figure 2
Schematic of one of the 20 achromats of the MAX IV 3 GeV storage ring. Magnets indicated are gradient dipoles (blue), focusing quadrupoles (red), sextupoles (green) and octupoles (brown).

'Tune' or Phase Advance of a Storage Ring

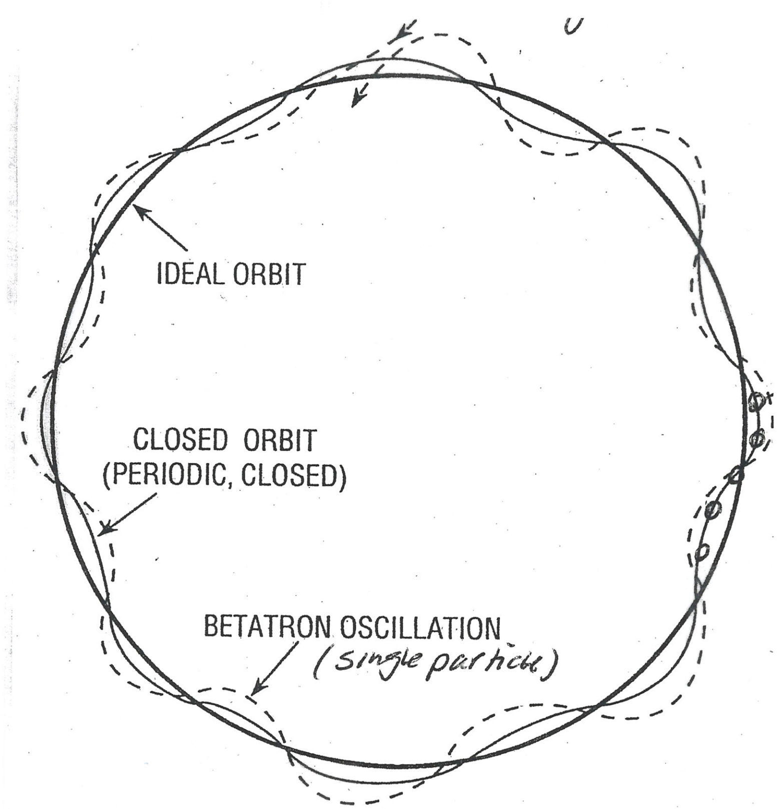
- Individual particles circulate with free betatron oscillations
- Phase advance per turn is called the 'tune'
- Working point (Q_x, Q_y)

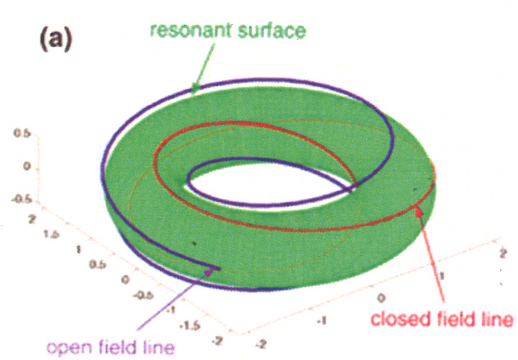
$$Q_x \sim 30.15 \qquad Q_x = \int \frac{ds}{\beta(s)}$$

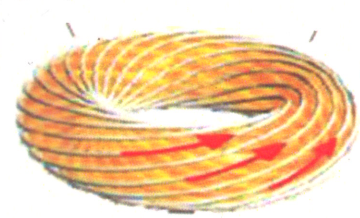
integer tune (30) — fractional tune (.15)

- Want fractional tune ~ irrational to avoid resonances (repeated driven motion)

$$m Q_x + n Q_y = p \qquad (m, n, p) \text{ integers}$$



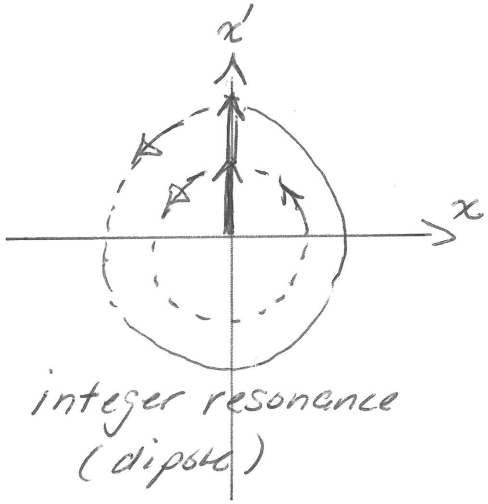




Resonances

numerology $mQ_x + nQ_y = p$ $(m, n, p) \pm$ integers

- sum resonances
- difference resonances
- stopbands
- driven by field errors in lattice

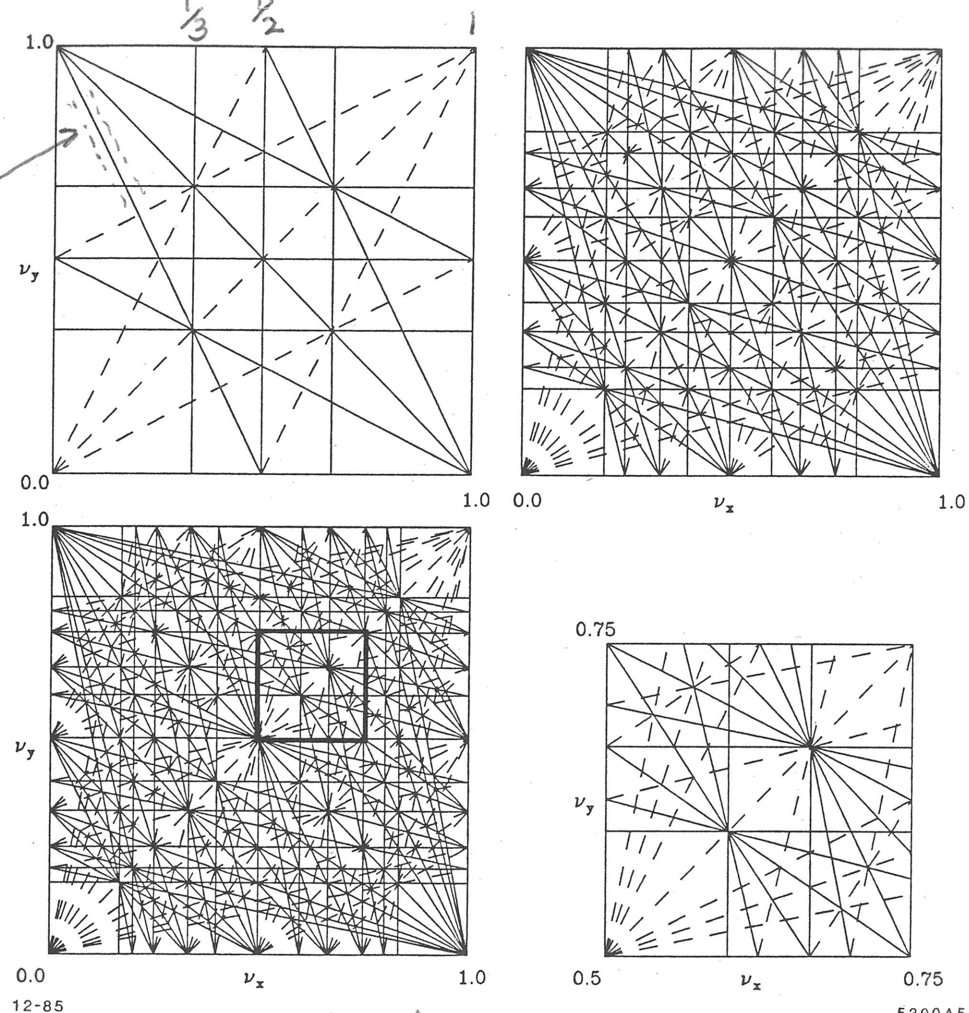


integer resonance (dipole)

$\frac{1}{2}$ - integer (quad)

$\frac{1}{3}$ - integer (sext)

⋮



Working Diagram

Fig. 5 Resonance lines in tune space.

$Q_x = \nu_x$

$Q_y = \nu_y$

"Avoid resonant tunes"

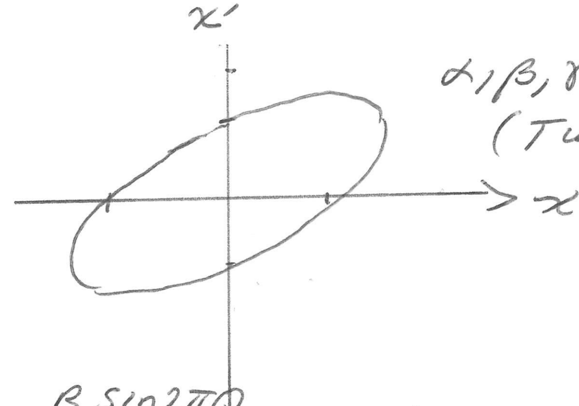
The One-Turn Transport Matrix

From before

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = M \begin{pmatrix} x \\ x' \end{pmatrix}_0 = \underbrace{\begin{pmatrix} \sqrt{\beta_1/\beta_0}\cos\Delta\phi + \alpha_0 \sin\Delta\phi & \sqrt{\beta_1/\beta_2}\,\sin\Delta\phi \\ \dfrac{-(1-\alpha_0\alpha_1)\sin\Delta\phi + (\alpha_1-\alpha_0)\cos\Delta\phi}{\sqrt{\beta_1/\beta_0}} & \sqrt{\beta_0/\beta_1}\left(\cos\Delta\phi - \alpha_1 \sin\Delta\phi\right) \end{pmatrix}}_{\text{General transport matrix } M} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

For a one-turn 'map'

$$\alpha_1 = \alpha_0 \qquad \beta_1 = \beta_0 \qquad \gamma_1 = \gamma_0$$



α, β, γ ellipse
(Twiss parameters)

$$\therefore \quad M_{rev} = \begin{pmatrix} \cos 2\pi Q + \alpha \sin 2\pi Q & \beta \sin 2\pi Q \\ -\dfrac{1-\alpha^2}{\beta}\sin 2\pi Q & \cos 2\pi Q - \alpha \sin 2\pi Q \end{pmatrix}$$

At a waist where $\alpha = 0$

$$\boxed{M_{rev} = \begin{pmatrix} \cos 2\pi Q & \beta \sin 2\pi Q \\ \dfrac{\sin 2\pi Q}{\beta} & \cos 2\pi Q \end{pmatrix}}$$

$$\boxed{\beta = \sqrt{\frac{m_{12}}{m_{21}}}}$$

$$\boxed{Q = \frac{1}{2\pi}\cos^{-1}\left(\frac{Tr(M)}{2}\right)}$$

Effect of Field Errors - Linear Tune Shift

$$M_{rev}^{New} = \begin{pmatrix} 1 & 1 \\ -\Delta KL \cdot & 0 \end{pmatrix} \circ M_{rev}^{Old} \qquad \Delta KL = \text{thin quad}$$

new tune

$$M_{rev}^{New} = \begin{pmatrix} 1 & 0 \\ -\Delta KL & 1 \end{pmatrix} \begin{pmatrix} \cos 2\pi Q & \beta \sin 2\pi Q \\ -\gamma \sin 2\pi Q & \cos 2\pi Q \end{pmatrix} \stackrel{\Delta}{=} \begin{pmatrix} \cos 2\pi Q' & \beta \sin 2\pi Q' \\ -\gamma \sin 2\pi Q' & \cos 2\pi Q' \end{pmatrix}$$

$Q' = Q + \Delta Q$

Same trace

$$\rightarrow 2\cos 2\pi Q - \underbrace{(\Delta KL)\beta}_{\text{"}K \cdot \beta\text{"}} \sin 2\pi Q = 2\cos 2\pi (Q + \Delta Q)$$

$\cos 2\pi \Delta Q \sim 1$

$\sin 2\pi \Delta Q \sim 2\pi \Delta Q$

$\Delta KL \cdot \beta = 4\pi \Delta Q$

$$\boxed{\Delta Q = \frac{1}{4\pi} (\Delta KL) \cdot \beta}$$

Large β, large effect

move tune across resonance

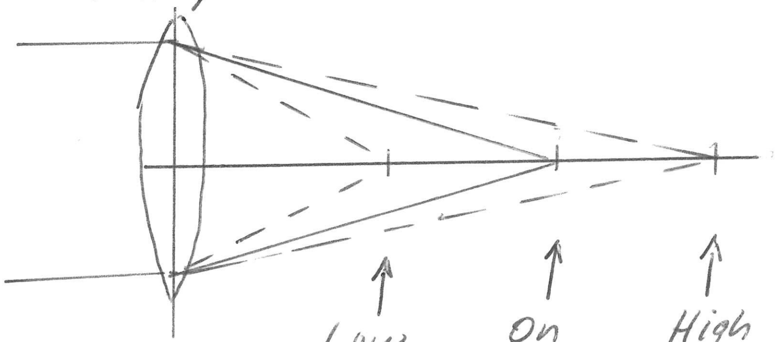
1) Change a magnet ΔKL

2) measure ΔQ

3) calculate β

Australian Synchrotron 2006

Chromaticity (tune shift with energy)



Quad strength for off-energy beam:

$$K_0 = \frac{\nabla B}{B\rho}$$

$$K_0 + \Delta K = K_0\left(1 + \frac{\Delta p}{p}\right)$$

$$\boxed{\Delta K = K_0 \cdot \frac{\Delta p}{p}}$$

$$\Delta Q = \frac{1}{4\pi}(\Delta K L)\cdot \beta$$

$$\Delta Q = \frac{1}{4\pi}\oint \Delta K \beta \, ds$$

$$\Delta Q = \frac{1}{4\pi}\frac{\Delta p}{p}\oint K\beta \, ds$$

$$\boxed{\xi \triangleq \frac{\Delta Q}{\frac{\Delta p}{p}} = \frac{1}{4\pi}\oint K(s)\beta(s)\, ds}$$

Example

20 cells

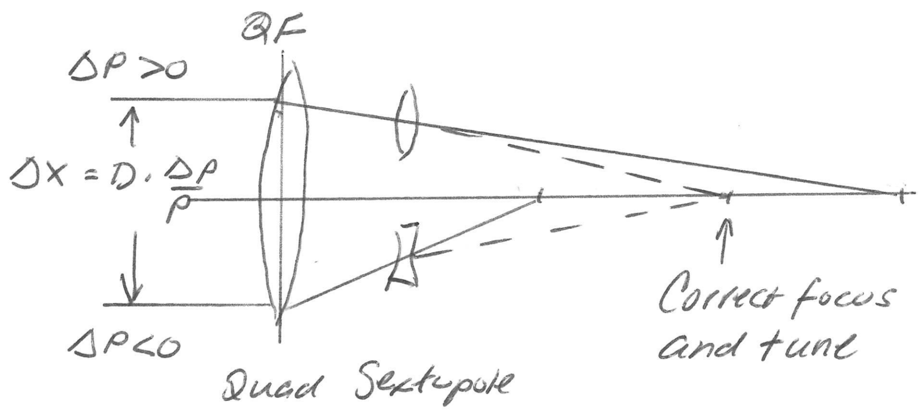
5 QF quads $KL = 1$ $\beta_x = 10$

5 QD quads $KL = -1$ $\beta_y = 5$

$$\xi_x = \frac{1}{4\pi}\left[20 \cdot 5 \cdot (10 - 5)\right]$$

$$\xi_x = \frac{1}{4\pi} \cdot 500 \sim \underline{\underline{50}} \text{ (minus)}$$

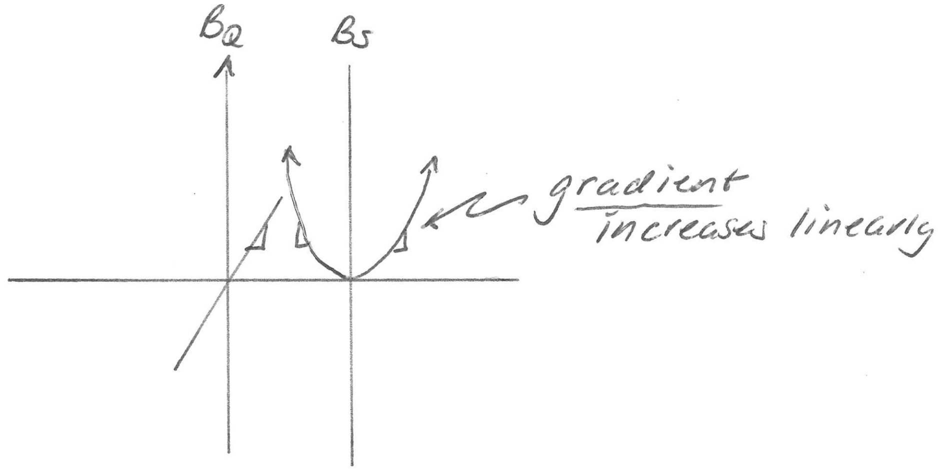
Chromaticity Correction (Head-Tail Instability)



For a sextupole $\Delta k = S_0 \left(D \cdot \frac{\Delta P}{P}\right)$

sextupol strength

$B = \frac{1}{2} S_0 x^2$



$$\Delta Q = \frac{1}{4\pi} \oint \Delta k \beta \, ds$$

$$\Delta Q = \frac{1}{4\pi} \oint \left[K_0 \frac{\Delta P}{P} + S_0 \cdot D \cdot \frac{\Delta P}{P} \right] \beta \, ds$$

$$\boxed{\xi = \frac{\Delta Q}{\frac{\Delta P}{P}} = \frac{1}{4\pi} \oint \left[K_0(s) + S_0(s) D(s) \right] \beta(s) \, ds}$$

The Billion Euro Question

$$\xi_{x,y} = \frac{1}{4\pi}\oint\left[K(s) + S(s)D(s)\right]\beta(s)\,ds \gtrsim 0 \quad \text{for beam stability}$$

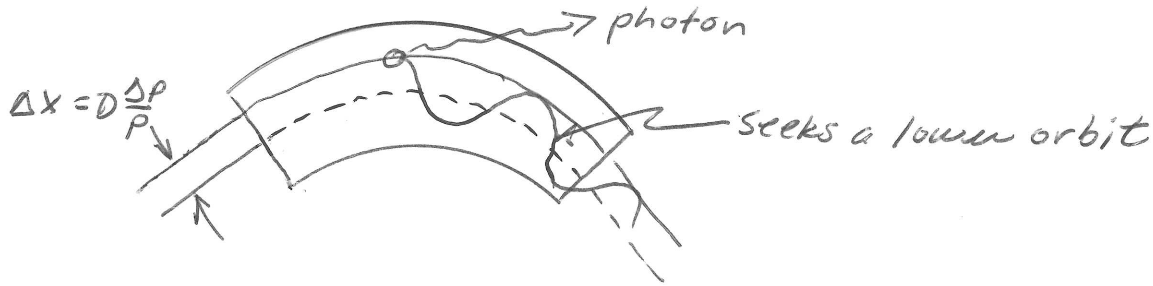
Strong quadrupoles for tight focus

Modest sextupoles for stable single particle motion

Low dispersion for small beam emittance

Brightness $B \cong \frac{\text{photon}}{\text{sec}} \cdot \underbrace{\frac{1}{\sigma_x} \cdot \frac{1}{\sigma_{x'}}}_{\text{emittance}} \cdot \frac{1}{BW}$

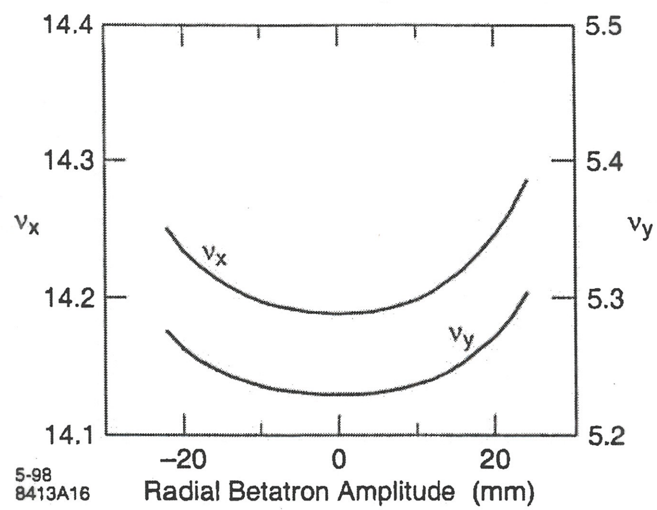
Similer to QF/QD quadrupoles, SF/SD tend to cancel
so need to operate with strong fields

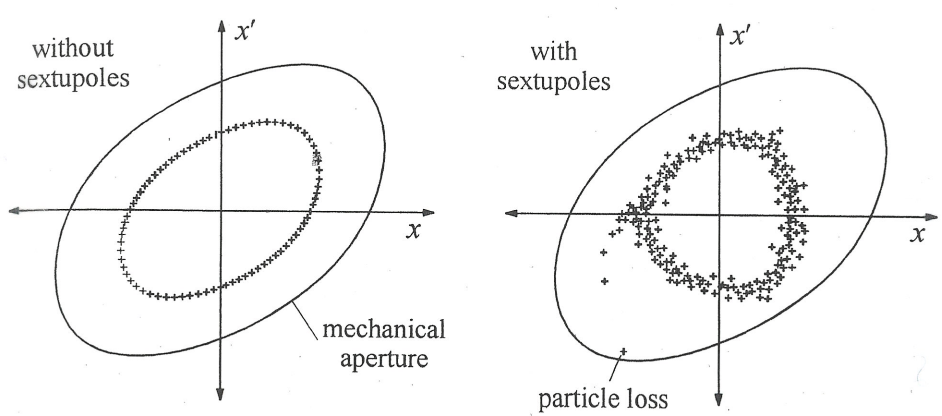


Resonances and 'Dynamic Aperture'

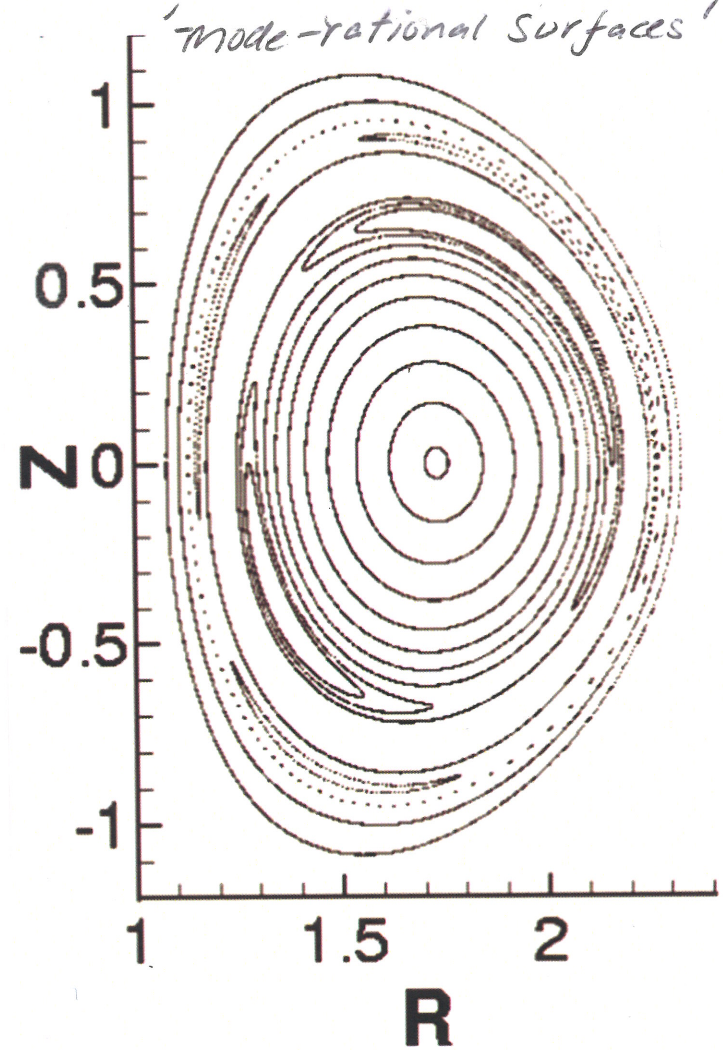
Strong sextupoles introduce non-linear fields

→ tune shift with betatron oscillation amplitude



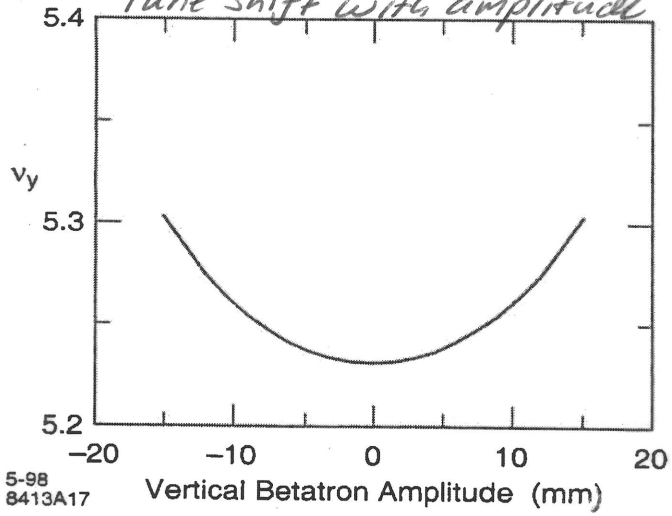


'mode-rational surfaces'

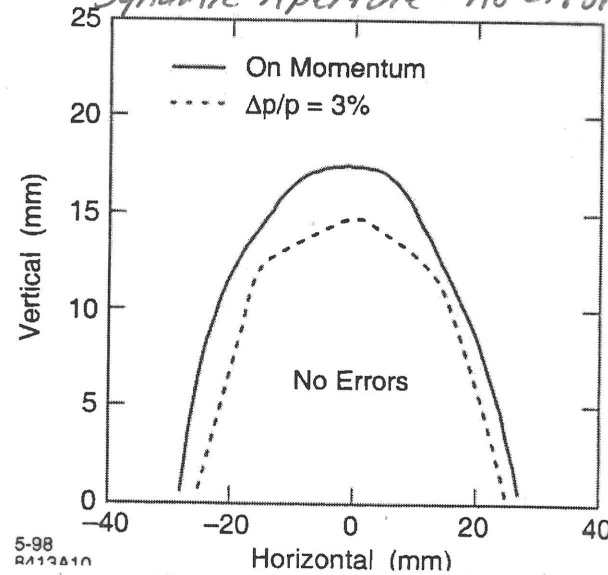


Dynamic Aperture

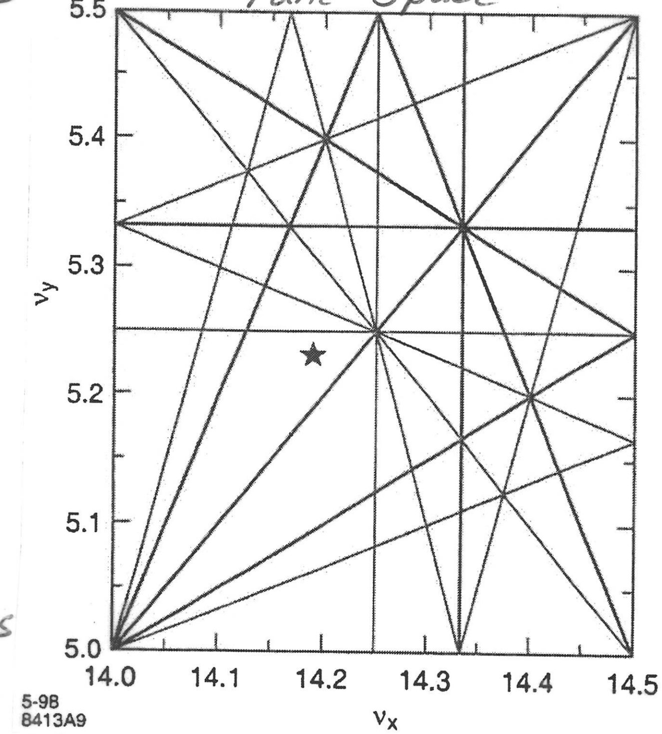
tune shift with amplitude



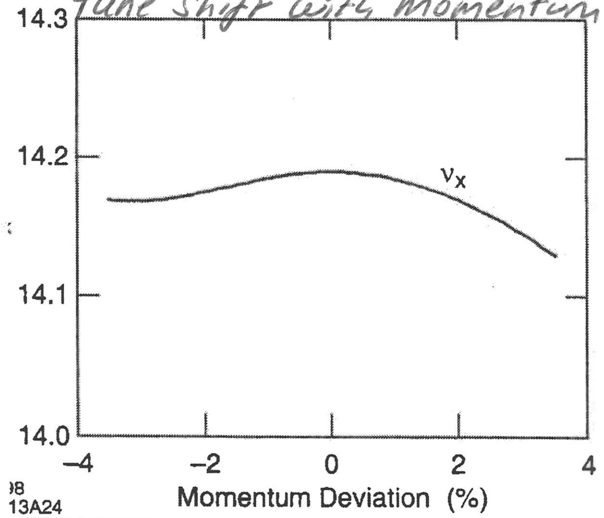
Dynamic Aperture - no errors



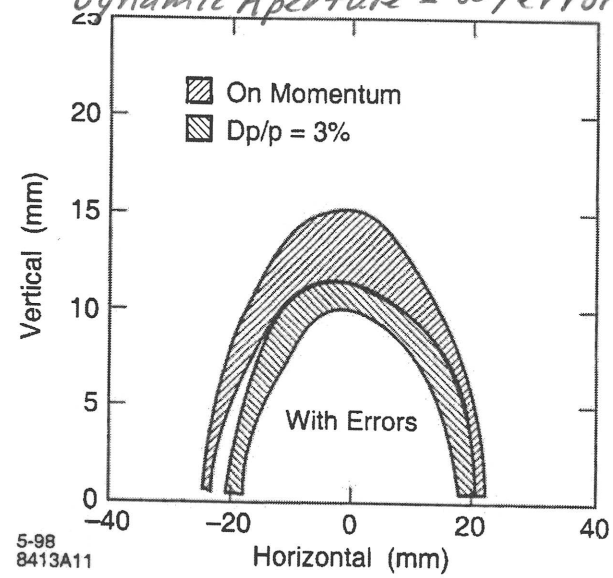
Tune Space



tune shift with momentum



Dynamic Aperture - w/errors



- capture efficiency
- beam lifetime

Floquet Transformation

Solution to $x'' + k^2(s)\,x = 0$

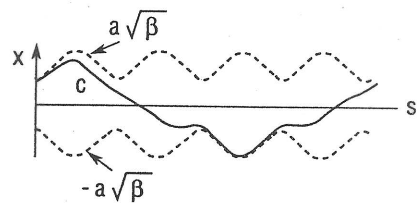
is $x(s) = A\sqrt{\beta(s)}\,\cos\phi(s)$ $\qquad \phi(s) = \int \frac{ds}{\beta}$

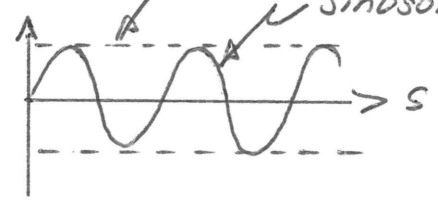
1) Normalized by $\sqrt{\beta}$ (Floquet transform)

$$\eta(s) = \frac{x(s)}{\sqrt{\beta}} = A\cos\phi(s)$$

2) Divide phase by tune

$$\Phi(s) = \frac{1}{Q}\int\frac{ds}{\beta} \qquad [0, 2\pi]$$



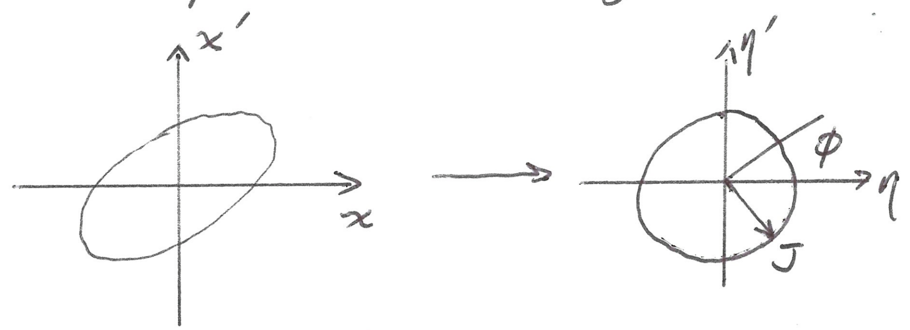


3) Resonance calculations $\quad \frac{d^2\eta}{d\Phi^2} + Q^2\eta = \underbrace{B_0 + B' + B'' + B''' + \cdots \text{HOT}}_{\text{driving terms}} \qquad ' \triangleq \frac{\partial}{\partial\eta}$

Floquet Transformation

- normalize by $\frac{1}{\sqrt{\beta}}$, $\frac{1}{Q}$

- Enters realm of Hamiltonian Dynamics



(J, ϕ) Action-angle

- Perturbation theory
  - tune shift with amplitude
  - resonances
  - Stop-band widths

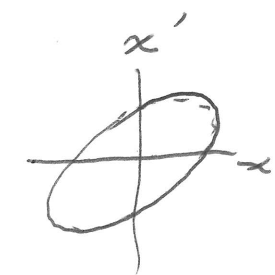
Review I

- Betafunctions define the beam envelope : $X(s) = \sqrt{\epsilon\beta(s)} \cos\phi(s)$

  $\sigma(s) = \sqrt{\epsilon\beta(s)}$

- Beam envelope is known in optical elements and between optical elements

- Twiss parameters (α, β, γ) define phase space ellipse

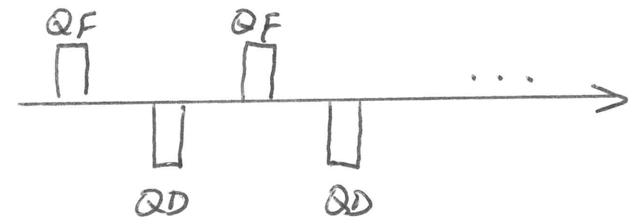


- Ellipse propagates as $\Sigma_1 = M \Sigma_0 M^T$ $\quad \Sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$

  $X_1 = M X_0$

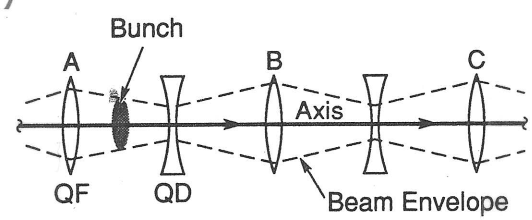
Review II

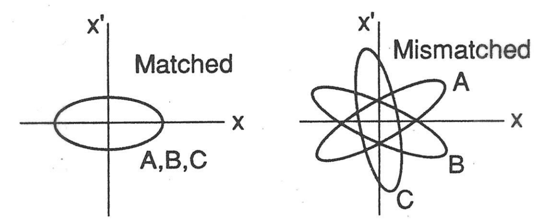
- Betafunctions derived from non-linear D.E. $x'' + K^2(s)x = 0$



  - $K(s)$ is piecewise-continuous
  - $\beta(s)$ is continuous function

- Solution to D.E. requires boundary conditions
  - transport line (β_0, β_0')





  - storage rings and cells use closure condition

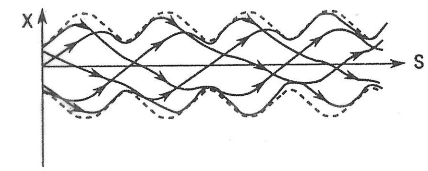
$$\beta_{start} = \beta_{end}$$

$$\beta'_{start} = \beta'_{end}$$

some hand calculations
many computers

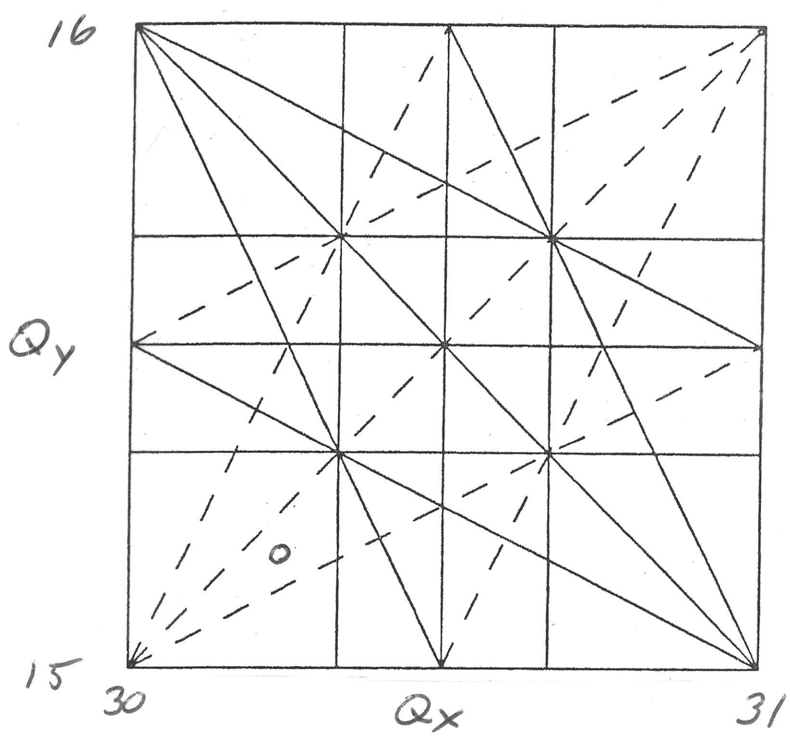
Review III

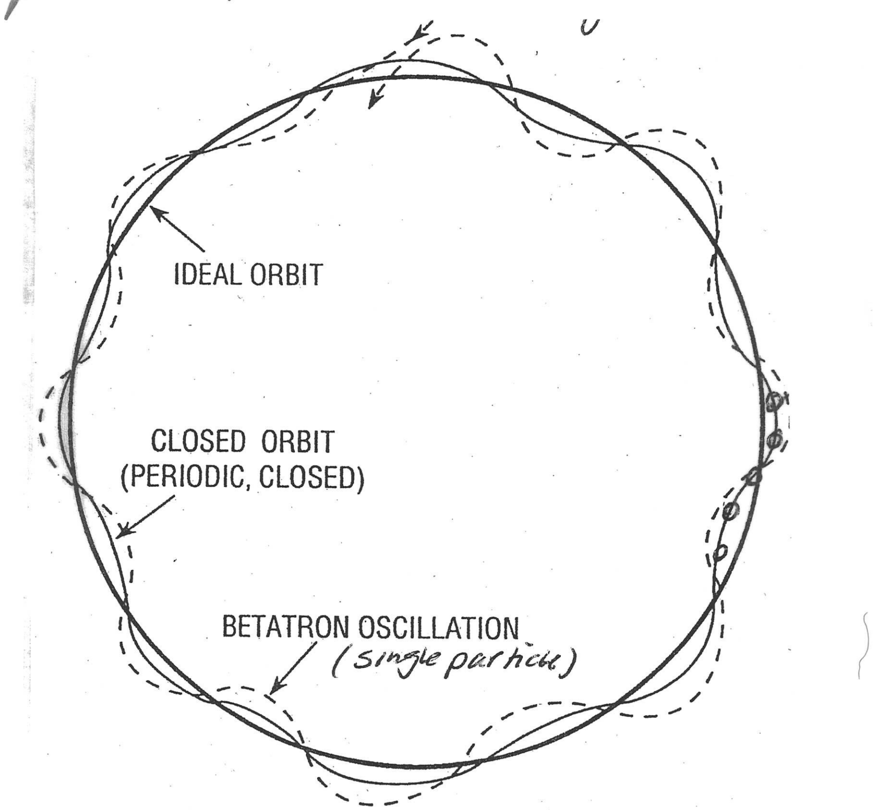
- Single particles move within the betatron envelope
  - Gaussian amplitude distribution
  - Random phase



- Sensitivity to field errors (avoid kicks at large β)

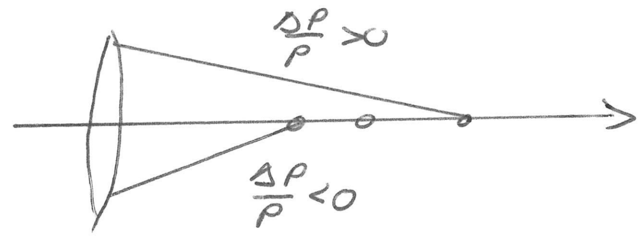
- In storage rings need non-integer phase advance to avoid resonances





Review IV

- Chromaticity changes phase advance with $\frac{\Delta p}{p}$



$K \to K_0 + \Delta K$ focusing change

$\beta \to \beta_0 + \Delta\beta$ beta-beats

$\phi \to \phi_0 + \Delta\phi$ chromaticity (tune crosses resonances)

- Use sextupoles to correct chromatic errors

non-linear fields generate geometric aberrations (vs. chromatic aberr.)

tune-shift with amplitude $\to$ resonances

loss of dynamic aperture

beam lifetime, injection