A Sidetrip to Fourier Series

I've made the claim the 1) if your circuit is linear and 2) you test it w/ a sinusoid, you know its response to any waveform. How do I know this? Fourier told me. He said that, for any periodic function:

(1) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are constant coefficients. This works since cos/sin are an orthogonal basis set.

Q: How to find a_n & b_n?

For a_n's, multiply both sides of (1) by $\cos mx$, integrate over cycle.

(2) $$\int_{-\pi}^{+\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_{-\pi}^{+\pi}\cos mx\,dx + \int_{-\pi}^{+\pi}\sum_{1}^{\infty} a_n \cos nx \cos mx\,dx + \int_{-\pi}^{+\pi}\sum_{1}^{\infty} b_n \cos mx \sin nx\,dx$$

Due to orthogonality, $$\int_{-\pi}^{+\pi} \cos nx \cos mx = 0 \quad n \neq m$$
$$= \pi \quad n = m$$

$\{ \; = \pi\delta_{mn}$ (see footnote)

$$\int_{-\pi}^{+\pi} \cos mx \sin nx = 0 \quad \text{always}$$

So 2 becomes
$$\int f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{+\pi}\overset{0}{\cos nx}\,dx + \pi a_n \qquad \left(\int \cos \text{ over a cycle} = 0 \right)$$

(3) or, $$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx$$

(4) similarly, $$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx$$

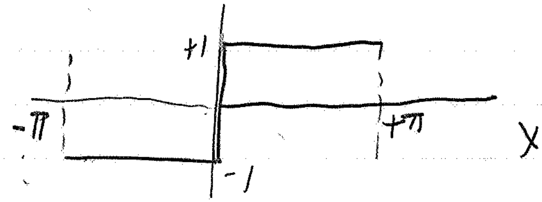
and, for $n=0$,
$$\int f(x)\cos(0)\,dx = \frac{a_0}{2}\int_{-\pi}^{+\pi}\cos(0)\,dx$$

(5) or $$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx$$

Footnote:
The "Kronecker Delta" has the property:
$\delta_{mn} = 1$ if m=n, otherwise = 0

Example: Square wave

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ +1 & 0 < x < \pi \\ 0 & x = 0 \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{\pi}\left[-\int_{-\pi}^{0} dx + \int_{0}^{\pi} dx\right] = \frac{-\pi + \pi}{\pi} = 0 = a_0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx$$

$$= \frac{1}{n\pi}\left[-\sin nx\Big|_{-\pi}^{0} + \sin nx\Big|_{0}^{\pi}\right] = 0 = a_n$$

$$b_n = \frac{1}{\pi}\int f(x)\sin nx\,dx = \frac{-1}{n\pi}\left[-\cos nx\Big|_{-\pi}^{0} + \cos nx\Big|_{0}^{\pi}\right]$$

$$= \frac{-1}{n\pi}\left[-2 + -2\right] = \frac{4}{n\pi} = b_n$$

so square wave =

$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty} \frac{1}{n}\sin nx \;, \text{ or } = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

$$= \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$