Accelerator Physics Lecture Modules

1. Introduction/Timing
2. Beam propagation basics
3. Simple Harmonic Oscillator

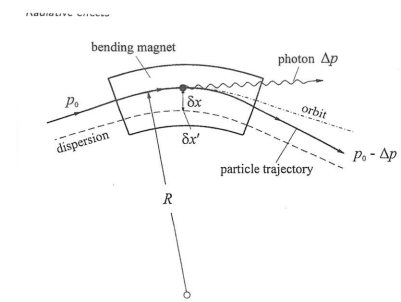
4. Betatron oscillations (transverse motion)
5. Dispersion
6. Closed Orbit Distortion

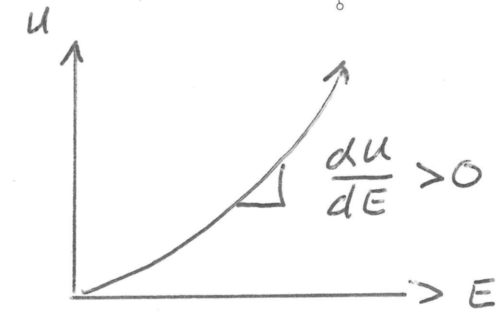
7. Synchrotron oscillations
8. Properties of synchrotron radiation

9. Radiation damping
10. Equilibirum emittance

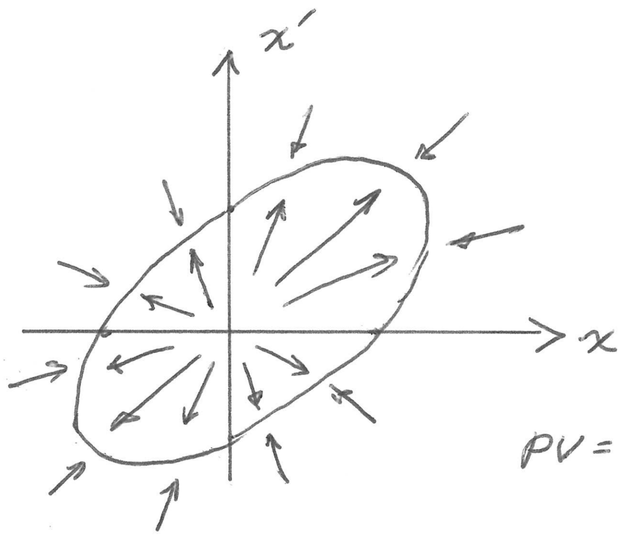
Equilibrium Emittance

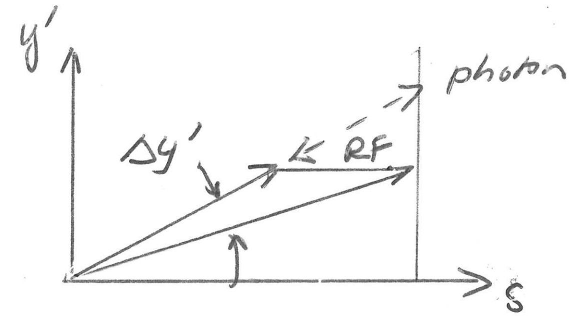
- Statistical rate of <u>quantum</u> <u>excitation</u>





Balanced by radiation damping





$PV = nRT$

Equilibrium emittance determined by magnet lattice: $I_1 - I_5$

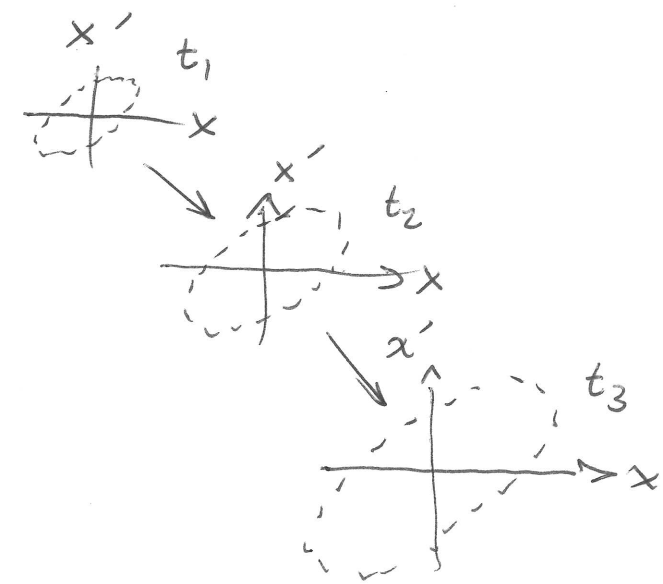
This is an integral component of "The Billion Dollar Question"

Particle Motion Excitation

- photon emission excites synchrotron and betatron oscillations

- random/stochastastic in nature
  - broad photon energy spectrum
  - emission at different values of $\begin{cases} \text{dispersion } D(s), D'(s) \\ \text{betafunction } \beta(s) \\ \text{bending radius } (ID\text{'s}) \end{cases}$
  - emission phases random (incoherent beam)

- Emission/excitation leads to blow-up (heating) of the beam
  - transverse emittance
  - longitudinal emittance
  - brightness
  - 200kW radiated power



- What keeps beam blow-up in check?

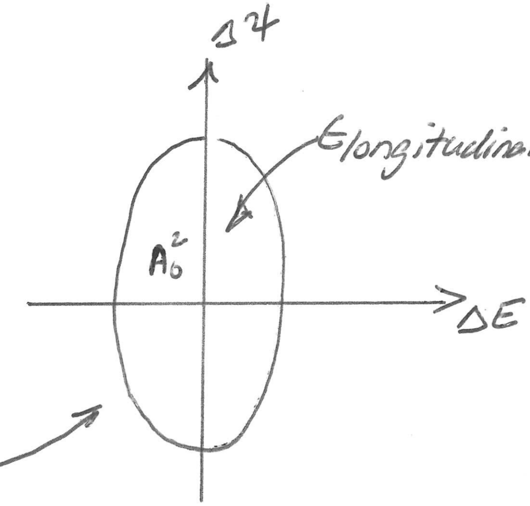
Start with Longitudinal Emittance

$\Delta E'' + \Omega_s^2 \Delta E = 0$ (undamped)

$\Delta E = A_0 \cos(\Omega_s t)$

$\Delta \psi = \left(\frac{h\omega_0 \alpha}{E\Omega_s}\right) A \sin(\Omega_s t)$

As before eliminate phase to get ellipse equation

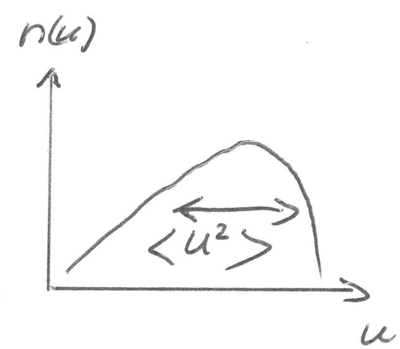


$$\boxed{\Delta E^2 + \left(\frac{E\Omega_s}{h\omega_0\alpha}\right)\Delta\psi^2 = A_0^2}$$

After photon emission $A = A_0 + \underset{\text{photon}}{u} \cos(\underset{\text{phase}}{\phi_{random}})$

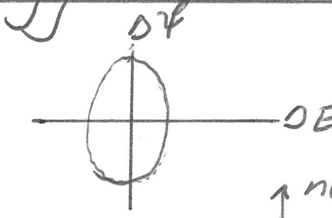
square $A^2 = A_0^2 + 2A_0 u \cos\phi_r + u^2 \cos^2\phi_r$

Average over many events : $\underset{\delta(\text{oscillation energy})}{\langle \delta A^2 \rangle} = \underset{\text{variance of photon spectrum}}{\langle u^2 \rangle}$

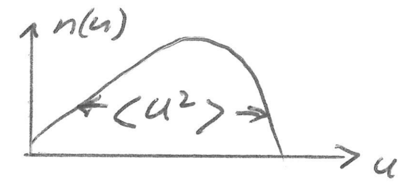


Equilibrium Energy Spread

$$\Delta E^2 + \left(\frac{E_0 \Omega_s}{h \omega_0 \alpha}\right)^2 \Delta \tau = A_0^2$$



$A = A_0 + u$ by photon emission



$$\langle \delta A^2 \rangle = \langle u^2 \rangle$$

$n(u)$ is number of photons/sec at energy u

Define $N_u = \int n(u)\,du$ = total number photons/sec

then $\dfrac{\langle u^2 \rangle}{\text{sec}} = \int u^2 n(u)\,du = N_u \langle u^2 \rangle$

Change of oscillation <u>energy</u> in time

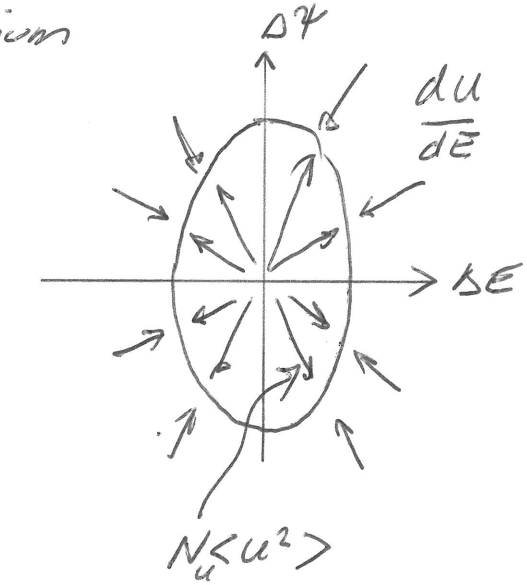
$$\boxed{\frac{d\langle \delta A^2 \rangle}{dt} = \frac{d}{dt}\langle u^2 \rangle = N_u \langle u^2 \rangle}$$

average growth rate for longitudinal energy oscillations due to photon emission

Equilibrium Energy Spread

Balance excitation and damping

$$\frac{d\langle \delta A^2\rangle}{dt} = \underbrace{N_u\langle u^2\rangle}_{\text{excitation from photon emission}} - 2\underbrace{\frac{\langle A^2\rangle}{\tau_s}}_{\text{radiation damping}} = \underbrace{0}_{\text{equilibrium}}$$



Solves $\langle A^2\rangle = \frac{\tau_s N_u\langle u^2\rangle}{2}$

$$\sigma_E^2 = \frac{\langle A^2\rangle}{2} = \frac{1}{4}\tau_s N_u\langle u^2\rangle \sim \frac{\tau_s}{4} u_c P_s$$

where $\tau_s = \frac{u_0}{T_0 E}$ $\quad u_c \propto E^3 \quad P_s \propto E^2$

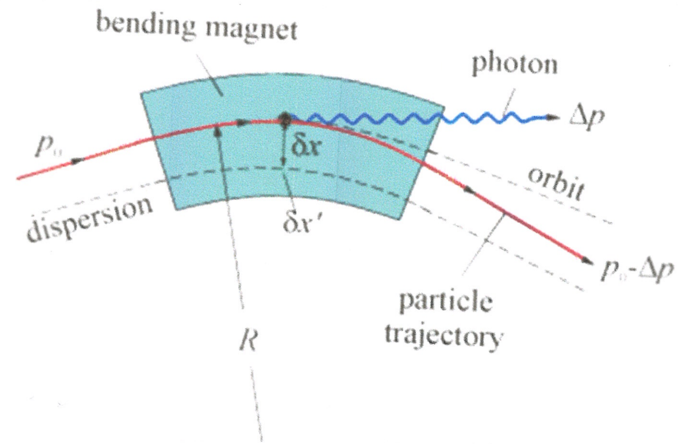
$$\sigma_E^2 \propto \frac{1}{E}\cdot E^3 \cdot E^2 = E^4$$

$$\left(\frac{\sigma_E}{E}\right)^2 \propto E^2$$ Equilibrium Energy Spread

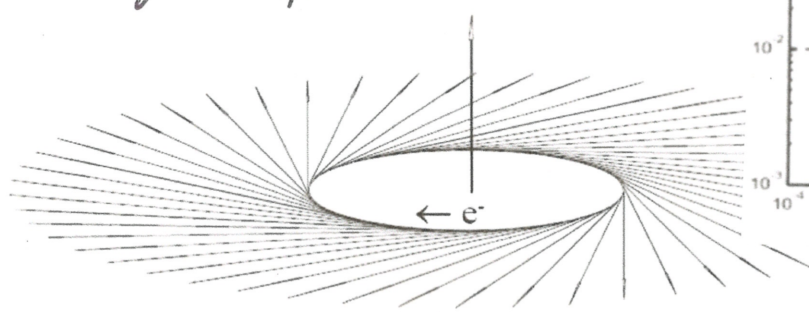
Horizontal Emittance

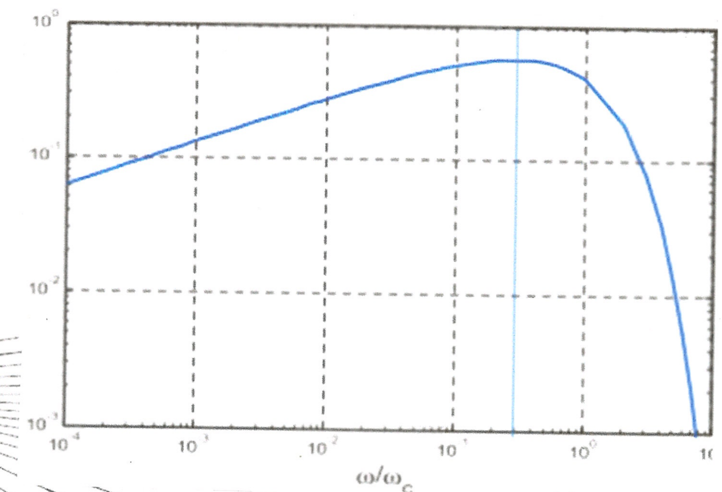
- Dipole magnets bend in the horizontal plane

1) Dispersion $x'' + \frac{1}{R^2}x = \frac{1}{R}\frac{\Delta p}{p}$



2) Broadband synchrotron radiation
- in the horizontal plane

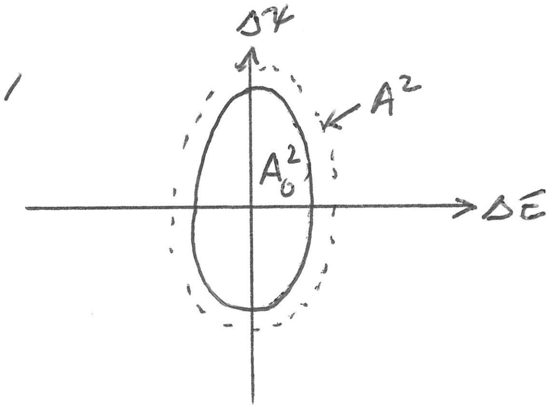




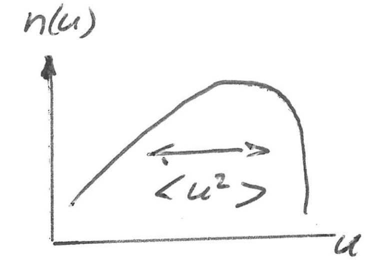
Horizontal Emittance

To calculate quantum excitation use similar phase-space approach

Longitudinal plane



$A = A_0 + u$ ← photon



$$\langle \delta A^2 \rangle = \langle u^2 \rangle$$

$$\frac{d\langle \delta A^2 \rangle}{dt} = N_u \langle u^2 \rangle$$

$$\frac{d\langle \delta A^2 \rangle}{dt} = \underbrace{N_u \langle u^2 \rangle}_{\text{excitation}} - \underbrace{\frac{\langle A^2 \rangle}{\tau_s}}_{\text{damping}} = 0$$

$$\boxed{\langle A^2 \rangle = \frac{\tau_s N_u \langle u^2 \rangle}{2}}$$

Equilibrium

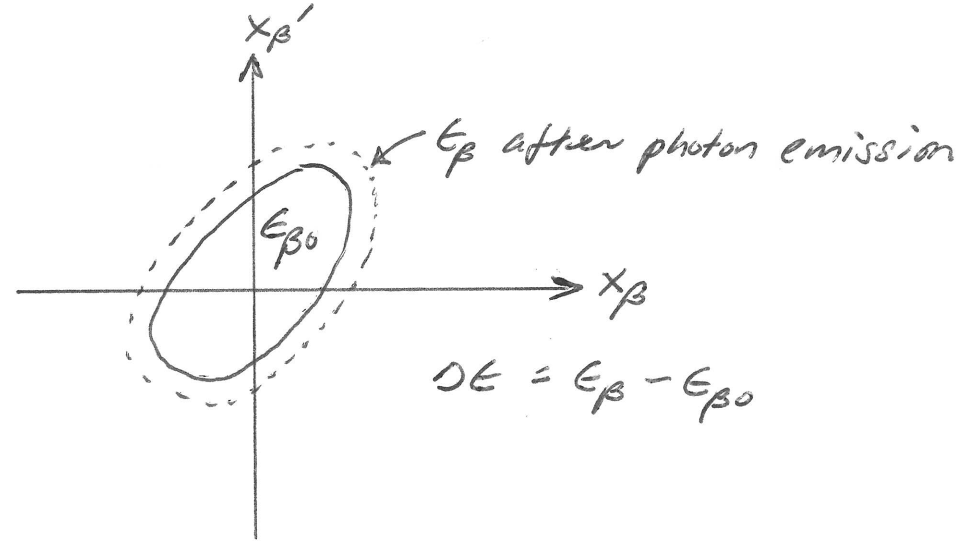
Horizontal Emittance

For betatron oscillations $\epsilon_{\beta 0} = \gamma x_\beta^2 + 2\alpha x_\beta x_\beta' + \beta x_\beta'^2$

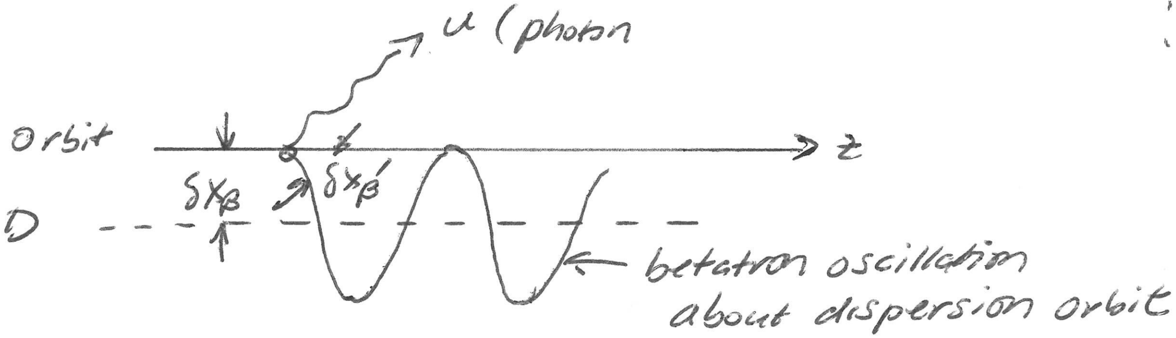
oscillation energy — betatron oscillation amplitude — slope

$\delta x_\beta = D \cdot \frac{u}{E}$ ← photon

$\delta x_\beta' = D' \cdot \frac{u}{E}$



$\Delta\epsilon = \epsilon_\beta - \epsilon_{\beta 0}$



Substitute $\begin{cases} x_\beta = x_{\beta 0} + \delta x_\beta \\ x_\beta' = x_{\beta 0}' + \delta x_\beta' \end{cases}$ in to ellipse equation

Horizontal Emittance

$$\epsilon_{\beta 0} = \gamma x_{\beta 0}^2 + 2\alpha x_{\beta 0} x'_{\beta 0} + \beta x'^2_{\beta 0}$$

$$\begin{cases} x_\beta = x_{\beta 0} + \delta x_\beta = x_{\beta 0} + D \frac{u}{E} \\ x'_\beta = x'_{\beta 0} + \delta x'_\beta = x'_{\beta 0} + D' \frac{u}{E} \end{cases}$$

Substitute

$$\delta \epsilon_\beta = \epsilon_\beta - \epsilon_{\beta 0} = \underbrace{\left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right)}_{\triangleq \mathcal{H}(s)} \cdot \frac{u^2}{E^2} - \underbrace{\left(\text{terms with } x_{\beta 0}, x'_{\beta 0}\right)}_{\substack{\text{average to zero} \\ \langle \cos \rangle = \langle \sin \rangle = 0}}$$

Average over betatron phases $x_\beta = \sqrt{\epsilon_\beta} \cos \phi$

$$\boxed{\delta \epsilon_\beta = \mathcal{H}(s) \cdot \frac{u^2}{E^2}}$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

average change in oscillation energy

Horizontal Emittance

$$\delta\epsilon_\beta = \mathcal{H}(s)\frac{u^2}{E^2}$$ for photon energy 'u' at position $\mathcal{H}(s)$

$$\mathcal{H}(s) = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

Average time-rate-of-change over many events

$$\frac{d\langle\delta\epsilon_\beta\rangle}{dt} = \int \frac{u^2}{E^2}\mathcal{H}(s)\, n(u)\, du$$

$$\frac{d}{dt}\langle\delta\epsilon_\beta\rangle = \mathcal{H}(s)\cdot N_u \frac{\langle u^2\rangle}{E^2}$$

$N_u = \int n(u)\, du$

= total photons/sec

$\mathcal{H}(s)$ term for betatron oscillations

(no $\mathcal{H}(s)$ for synchrotron oscillations)

Average over one turn $\int \mathcal{H}(s)\, ds$

$$\frac{d}{dt}\langle\delta\epsilon_\beta\rangle = \langle\mathcal{H}\rangle \cdot N_u \cdot \frac{\langle u^2\rangle}{E^2}$$

not quite correct, only contributes where photons emitted

Horizontal Emittance

For longitudinal motion $\qquad \frac{d}{dt}\langle \delta A^2 \rangle = N_u \langle u^2 \rangle$

For betatron motion $\qquad \frac{d}{dt}\langle \delta \epsilon_\beta \rangle = \langle \mathcal{H} \rangle N_u \cdot \frac{\langle u^2 \rangle}{E^2}$.

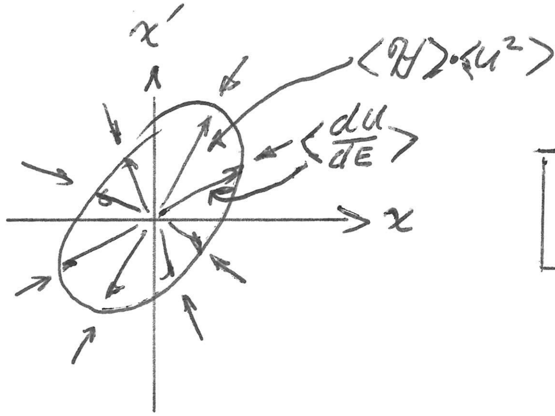
where $\mathcal{H}(s) = \gamma D^2 + 2\alpha D D' + \beta D'^2$

Solve for equilibrium state

beam emittance

$$\frac{d}{dt}\langle \delta \epsilon_\beta \rangle = 0 = \underbrace{\langle \mathcal{H} \rangle N_u \frac{\langle u^2 \rangle}{E^2}}_{\text{excitation}} - \underbrace{\frac{2\langle \epsilon \rangle}{\tau_x}}_{\text{damping}}$$

damping time



$$\boxed{\langle \epsilon_x \rangle = \langle \mathcal{H} \rangle \cdot N_u \frac{\langle u^2 \rangle}{E^2} \cdot \frac{\tau_x}{2}}$$ (approximately)

dispersion term
differs from longitudinal calculation

$$\sigma_x = \sqrt{\epsilon_x \beta_x} \longrightarrow \sqrt{\epsilon_x \beta_x + D^2 \cdot \left(\frac{\Delta p}{p}\right)^2}$$

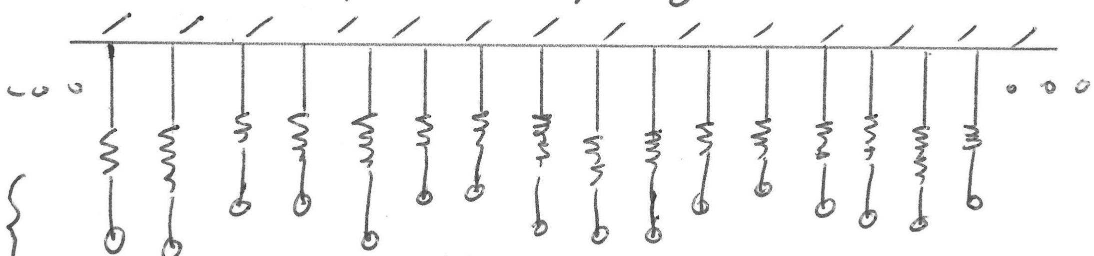
Review of Emittance Calculation

Equilibrium emittance is a balance between

1) quantum excitation

2) radiation damping

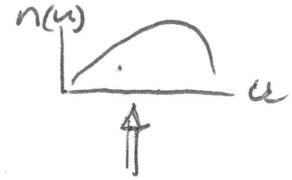
Ensemble of mass-spring oscillators



$\langle x^2\rangle, \langle x'^2\rangle$

$x'' + 2\alpha x' + \omega^2 x = 0$

damping



random kicks (amplitude and time)

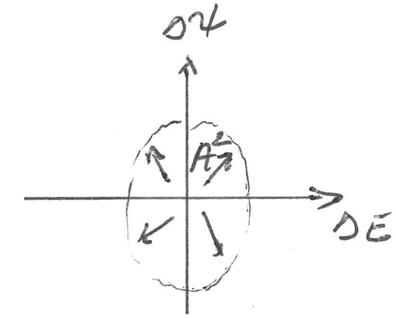
Balance: $\dfrac{\langle x^2\rangle}{\tau_{damp}} \sim N_u \langle u^2\rangle \Rightarrow \langle x^2\rangle = N_u \cdot \langle u^2\rangle \cdot \tau_d$

Emittance Calculation Review

Balance: quantum excitation = radiation damping

1) quantum excitation

o Photon emission stimulates synchrotron oscillations



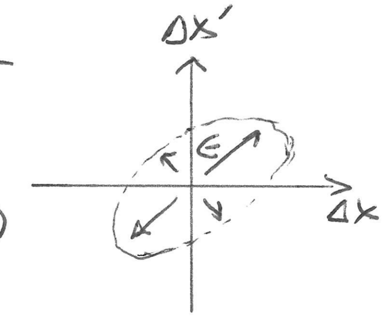
ΔE from photon

$\Delta\psi$ from change of path length (α)

RF restoring force

$$\frac{d}{dt}\langle \delta A^2 \rangle = N_u \langle u^2 \rangle$$

o Photon emission stimulates betatron oscillations



$\Delta x, \Delta x'$ from photon emission (D, D')

Quadrupole restoring force $K(s) \longrightarrow \beta(s)$

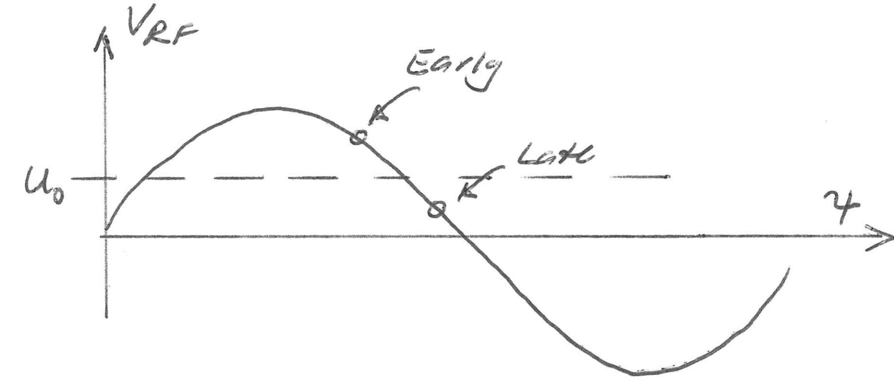
$$\frac{d}{dt}\langle \epsilon \rangle = \langle \mathcal{H} \rangle \cdot N_u \cdot \langle u^2 \rangle$$

Vertical oscillations from
betatron coupling
dispersion coupling
finite radiation opening angle (quantum limit)

Emittance Calculation

Balance: quantum excitation = radiation damping

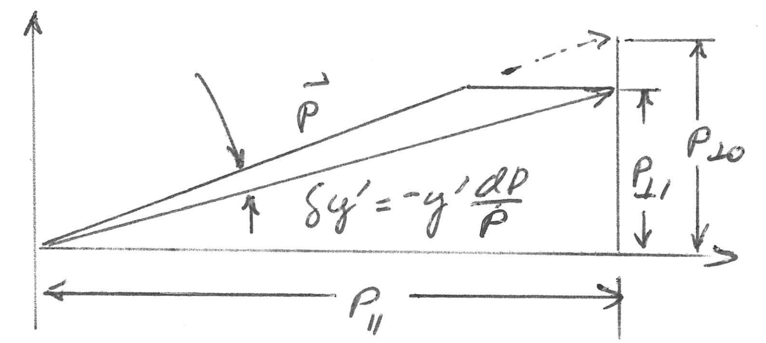
2) Radiation Damping



- Synchrotron oscillations

$$\frac{dU}{dE} > 0$$

$$\frac{\langle A^2 \rangle}{\tau_s}$$



- Betatron Damping

$$\frac{\langle \epsilon \rangle}{\tau_\beta}$$

Equilibrium Emittance

	Quantum Excitation	= Radiation Damping	Equilibrium	
Synchrotron	$N_u \langle u^2 \rangle$	$\dfrac{2\langle A^2 \rangle}{\tau_s}$	$\langle A^2 \rangle = \dfrac{N_u \langle u^2 \rangle \tau_s}{2}$	(ϵ, τ)
Betatron	$\langle \mathcal{H} \rangle N_u \langle u^2 \rangle$	$\dfrac{2\langle \epsilon \rangle}{\tau_x}$	$\langle \epsilon \rangle = \langle \mathcal{H} \rangle \dfrac{N_u \langle u^2 \rangle \tau_x}{2E^2}$	(x, x')

$$\alpha_s = \frac{U_0}{2T_0 E}(2 + \mathcal{D})$$

$$\alpha_x = \frac{U_0}{2T_0 E}(1 - \mathcal{D})$$

$$N_u = \int n(u)\,du$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

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